Geometric flow：

1. Mean curvature flow $\frac{\partial X}{\partial t} = -Hn$

2. Ricci flow $\frac{\partial g}{\partial t} = -2Ric(g)$

3. Yamabe flow $\frac{\partial g}{\partial t} = -Rg$

4. Harmonic map flow

5. Gauss curvature flow $\frac{\partial X}{\partial t} = -Kn$

§ 01 Ricci curvature

Ricci curvature provides a way to measure how the volume of a small geodesic ball in a curved space differs from that in Euclidean space。

The Ricci curvature is a tensor derived from the Riemann curvature tensor。Given a Riemannian manifold (M,g) with the Levi-Civita connection ∇，the Riemann curvature tensor $R(X,Y)Z$ measures how tangent vectors change under parallel transport around an infinitesimal loop。

The Ricci curvature tensor Ric is obtained by taking a trace(contraction) of the Riemann curvature tensor： $Ric(Y,Z) = \sum_{i=1}^{n} R(e_i, Y, Z, e_i)$

where $\{e_i\}$ is an orthonormal basis for the tangent space。

In local coordinates，the Ricci tensor is given by： $R_{ij} = R_{ikj}^{k}$

where R_{ikj}^{k} are the components of the Riemann curvature tensor。

§ 02 Ricci flow Richard Hamilton 1982

We have a Riemannian manifold M with the metric g_0，the Ricci flow is a PDE that evolves the metric tensor： $\frac{\partial}{\partial t} g(t) = -2Ric(g(t))$ ， $g(0) = g_0$

A solution to this equation (or a Ricci flow) is a one-parameter family of metrics g(t)，$(\mathrm{M}, g(\mathrm{t}_0))$ is called the initial condition (or initial metric)。

We hope that the metric will evolve towards one of the Thurston eight fundamental geometric structure。

The space-time for a Ricci flow is $M \times I$，where $t \in I$。
Given (p，t) and r>0，B(p，t，r) is the ball of radius r centered at (p，t) in the t time-slice。

S^n (n>1) of radius r(t)，the metric is given $g = r^2 \tilde{g}$，where $\tilde{g}$ is the metric on the unit sphere。
Since Ric(g)=(n-1)g，the Ricci flow becomes a ODE。

$$\frac{\partial}{\partial t}(r^2 \tilde{g}) = -2(n-1)\tilde{g}$$

$$\frac{dr^2}{dt} = -2(n-1)$$

$$r^2 = R_0^2 - 2(n-1)t$$

$r(t) = \sqrt{R_0^2 - 2(n-1)t}$，the sphere shrinks to a point as $t \to \dfrac{R_0^2}{2(n-1)}$。

§ 03 Ricci solton

A Ricci soliton is a special solution to the Ricci flow，a geometric flow that evolves a Riemannian metric on a manifold。

A Riemannian metric g on a manifold M is called a Ricci soliton if there exists a smooth vector field X on M and a constant $\lambda \in \mathrm{R}$ such that： $\mathrm{R}ic(g) + \frac{1}{2} L_X g = \lambda g$

§ 04 Einstein equation $\quad Ric(g) - \frac{1}{2} R_g = 8\pi T$

EVE(Einstein Vacuum equation 愛因斯坦真空方程) Ric(g)=0 $H(\Sigma) = 0$

Einstein manifolds： $Ric(g)$=λg

§ 05 Black hole

(1) Schwarzschild solution 1915

(2) Kerr solution 1968

(3) Penrose singularity theorem 1965

(4) No-hair theorem Werner Israel 1965

發現重力波 2015 事件視界望遠鏡 2022

EMRI(Extreme Mass Ratio Inspirals)：小黑洞繞這超大質量黑洞旋轉。