§ 1-1 平行移動(parallel transport)

流形上的微分有三種：(1)外微分 (2)李導數 (3)協變微分

向量場的協變微分 $\frac{Dw}{dt}=(\frac{dw}{dt})^T$

協變微分的幾何意義是：利用「Y 沿 X 平行移動」代替李導數中「Y 沿 X 的流線」來疊合不同點的切向量，從而引入對切向場的微分。

協變微分與平行移動互相決定。

設曲面 M 上一條曲線$\gamma=\gamma(t)$上有一個向量場 V，設 X=$\frac{d\gamma}{dt}$表示$\gamma(t)$的切向量，

若$\nabla_X V=0$ 則稱 V 是沿γ的平行場。

1. $\nabla_X Y=\sum_i (XY^i+\sum_{j,k}\Gamma^i_{jk}X^jY^k)\frac{\partial}{\partial x^i}$
2. 平行移動(parallel transport) $\dot{V}^i+\sum_{j,k}\Gamma^i_{jk}\dot{x}^jV^k=0$

   一個 vector field V^μ 沿曲線 $x^\mu(\lambda)$ 平行移動則 $\frac{d}{d\lambda}V^\mu+\Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho=0$

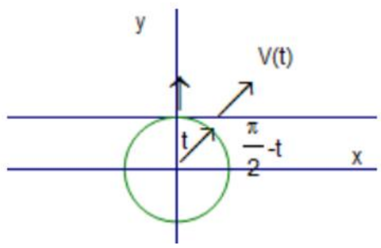
   A smooth vector field V(t) along c is parallel if $\frac{DV}{dt}\equiv 0$ on I 。

   $\frac{DV}{dt}=0 \Leftrightarrow v^i+\sum_{j=1}^{n}\omega^i_j(c'(t))v^j=0$ for i=1,…,n
3. $\ddot{y}+k^2y=0$ 則 y=A cos(k t)+ B sin(k t)
4. $\Gamma^k_{ij}=\frac{1}{2}g^{kl}\{\frac{\partial g_{jl}}{\partial x^i}+\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\}$

§ 1-2 雙曲平面 $H=\{(x,y)\in R^2 \mid y>0\}$ with the metric given by

$g_{11}=g_{22}=\frac{1}{y^2}, g_{12}=0$

(metric of Lobatchevski non-euclidean geometry)



$V(0)=(0,1), \gamma:\begin{cases}x=t\\y=1\end{cases}$

V(t)：v_0沿γ的平行移動

證明 V(t)與 y 軸方向夾角= t

v_0 is the tangent vector at the point (0,1)

$\Gamma_{11}^{1}=\Gamma_{12}^{2}=\Gamma_{22}^{1}=0,\Gamma_{11}^{2}=\frac{1}{y},\Gamma_{22}^{2}=\Gamma_{12}^{1}=-\frac{1}{y}$

在γ上$\Gamma_{11}^{2}=1,\Gamma_{22}^{2}=\Gamma_{12}^{1}=-1$，V(t)=(a(t),b(t))

$\frac{d}{dt}V^{\mu}+\Gamma_{\sigma\rho}^{\mu}\frac{dx^{\sigma}}{dt}V^{\rho}=0$

$\mu=1$時，$\frac{da(t)}{dt}-b(t)=0$，$\mu=2$時，$\frac{db(t)}{dt}+a(t)=0$

(for $\mu=2$，$\frac{db}{dt}+\Gamma_{11}^{2}\frac{dx^{1}}{dt}V^{1}=0$，where $x^{1}=t,V^{1}=a(t),\Gamma_{11}^{2}=1$)

$$\begin{cases}\frac{da}{dt}-b=0\ldots(1)\\ \frac{db}{dt}+a=0\ldots(2)\end{cases}$$

$\ddot{a}=\dot{b}=-a$，取$a=\cos\theta(t),b=\sin\theta(t)$，$V(t)=(\cos\theta(t),\sin\theta(t))$

$$\begin{cases}-\sin\theta\frac{d\theta}{dt}-\sin\theta=0\\ \cos\theta\frac{d\theta}{dt}+\cos\theta=0\end{cases}\text{所以}\frac{d\theta}{dt}=-1$$

$\theta=-t+c,V(0)=(\cos c,\sin c)=(0,1),c=\frac{\pi}{2},\theta=\frac{\pi}{2}-t$

即 V(t)與 y 軸方向夾角= t

§ 1-3

例 在球面上,$\gamma(t)=[\cos t,\sin t,0]$, $X=\frac{d\gamma}{dt}=[-\sin t,\cos t,0]$, $V=\frac{1}{5}[-\sin t,\cos t,1]$

則$\nabla_{X}V=\frac{DV}{dt}=(\frac{dV}{dt})^{T}=(-\frac{1}{5}[\cos t,\sin t,0])^{T}=0$ ($\frac{dV}{dt}\bullet X=0$，$\frac{dV}{dt}$在 X 方向的分量=0)，所以稱 V 沿赤道$\gamma(t)$平行

§ 1-4 球面S^{2}上的平行移動

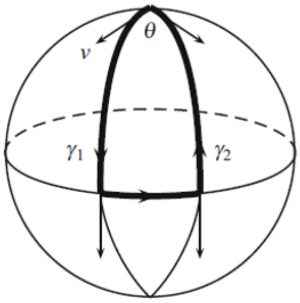


Fig. 14.5. Parallel translating *v* along a closed piecewise smooth geodesic.

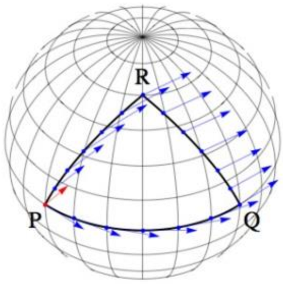
γ_1,γ_2是球面S^2上由北極 p 點出發的兩條經線(meridians)，兩者都是測地線，夾角為θ。

V 是在 p 點γ_1的切向量，讓 V 沿著γ_1平行移動到赤道，沿赤道平行移動到γ_2，

再沿γ_2平行移動到北極 p 點，此時的向量為 W。

則 W 與 V 的夾角為θ。

微分幾何中，一個微分流形上的聯絡的完整（英語：holonomy，又譯和樂），描述向量繞閉圈平行移動一周回到起點後，與原先相異的現象。



Parallel transport of a vector around a triangular path PQRP on the surface of a sphere。

The final vector will be rotated w.r.t.the initial vector。

The angle of rotation depends on the size of the loop，the path chosen，and the curvature of the manifold。

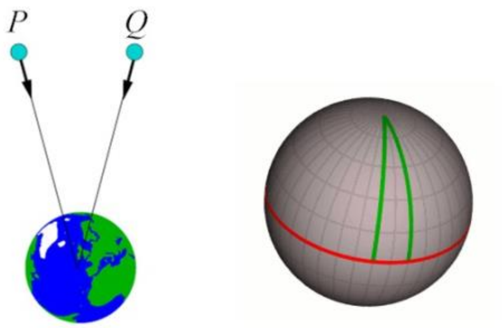
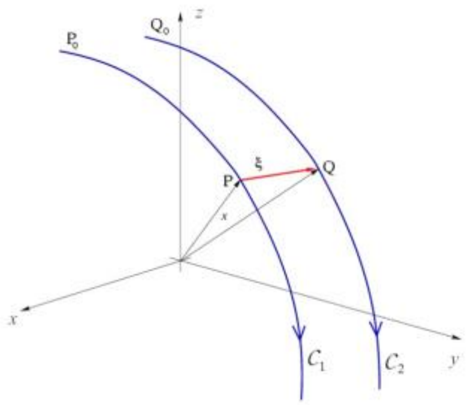


Figure 4: Left: two free-falling particles move along initially parallel paths towards the center of the Earth. There, both paths intersect; right: lines that are initially parallel on the surface of the Earth at the equator, intersect at the North pole.

P and Q，and we observe that both particles follow paths that lead to the center of the Earth。

From the perspective of the observer that is in free-fall with the particles，we see that the

particles move towards each other。
This is caused by the differential gravitational accerleration of the particles through what are called tidal forces。[tidal force]
According to Newton both paths interact because of gravitation，while according to Einstein this occurs because spacetime is curved。



What Newton calls gravitation is called curvature of spacetime by Einstein。
The trajectories of two free-falling particles in a gravitational field Ø 。
The three-vector $\vec{\xi}$ measures the distance between the two particles and is a fuction of time。
The Newtonian equations of the motion for particles P and Q are

$$\left(\frac{d^2x_j}{dt^2}\right)_{(P)} = -\left(\frac{\partial\Phi}{\partial x^j}\right)_{(P)} \quad \text{and} \quad \left(\frac{d^2x_j}{dt^2}\right)_{(Q)} = -\left(\frac{\partial\Phi}{\partial x^j}\right)_{(Q)}, \tag{1.42}$$

with Φ the gravitational potential. We define $\vec{\xi}$ as the separation between both particles. For parallel trajectories one has $\frac{d\vec{\xi}}{dt} = 0$. With $\vec{\xi} = (x_j)_{(P)} - (x_j)_{(Q)}$ we find from a Taylor expansion that to leading order in the small separation $\vec{\xi}$

$$\frac{d^2\xi_j}{dt^2} = -\left(\frac{\partial^2\Phi}{\partial x^j\partial x^k}\right)\xi_k = -\mathcal{E}_{jk}\xi_k \quad \rightarrow \quad \mathcal{E}_{jk} = \left(\frac{\partial^2\Phi}{\partial x^j\partial x^k}\right), \tag{1.43}$$

with $\mathcal{E}$ the gravitational tidal tensor. Notice that the metric for the 3D Euclidian space is given by $\delta_{jk} = \text{diag}(1,1,1)$ and that there is no difference between lower and upper indices. Eq. (1.43) is called the equation of Newtonian geodesic deviation.

According to Newton, particles moves towards each other and we write

$$\frac{d^2\vec{\xi}}{dt^2} = -\mathcal{E}(\_,\vec{\xi}) \tag{1.44}$$

in abstract notation. It is interesting that the field equation of Newtonian gravitation,

$$\nabla^2\Phi = 4\pi G\rho, \tag{1.45}$$

can be expressed in terms of second derivatives of Φ, which describe the tidal accelerations in Eq. (1.43). There is an analogous connection in GR.

註：

1. 平行性與共變微分相互決定 [微分幾何講稿]p.124~135
2. 大域微分幾何 p.156~162

當 $X\cdot<Y,Z>=<\nabla_X Y,Z>+<Y,\nabla_X Z>$ 則稱 ∇ 與 metric 相容

一個可微流形若有了協變微分，便稱為仿射流形(affine manifold)。

§ 1-5 Proposition

1. ∇與<.>相容⇔C 是 smooth curve X、Y 是沿 C 的平行向量場 則 <X,Y>=constant
2. On a Riemannian manifold M parallel translation preserves length and inner product：

If V(t) and W(t) are parallel vector field along a smooth curve $c:[a,b]\to M$ ，then the length $\|V(t)\|$ and the inner product $\langle V(t),W(t)\rangle$ are constant for all $t\in[a,b]$

杜武亮 p.112

Proof. Since $\|V(t)\| = \sqrt{\langle V(t),V(t)\rangle}$, it suffices to prove that $\langle V(t),W(t)\rangle$ is constant. By the product rule for the covariant derivative of a connection compatible with the metric (Theorem 13.2),

$$\frac{d}{dt}\langle V,W\rangle = \left\langle \frac{DV}{dt},W\right\rangle + \left\langle V,\frac{DW}{dt}\right\rangle = 0,$$

since $DV/dt = 0$ and $DW/dt = 0$. Thus $\langle V(t),W(t)\rangle$ is constant as a function of t. □

§ 1-6

Leon Foucault(1819~1868) Gregorio Ricci(1853~1925) Levi-Civita(1873~1941)

Leon Foucault(1819~1868)、Gregorio Ricci(1853~1925)、 Levi-Civita(1873~1941)

1851 年，法國科學家傅科(Leon Foucault 1819~1868)為了證明地球自轉做了一個傅科擺實驗。

鐘擺運動方向會隨時間改變方向(依鐘擺所在位置不同)。

若把鐘擺放在兩極，則每 24 小時繞一圈，若放在巴黎，則每天轉不到一圈。

傅科認為 鐘擺運動方向只是幻象，事實上它一直朝同一個方向擺動，實際上轉動的是地球。

但是絕對運動並不存在，地球轉動是相對於某一"不動"的東西而言，這不動的東西是甚麼？

科學家認為 傅科擺的行為不是以局部的環境為基準，而是以最遙遠的星系為準，與整個宇宙聯繫，這是佛法中所謂的"現象相依性"。

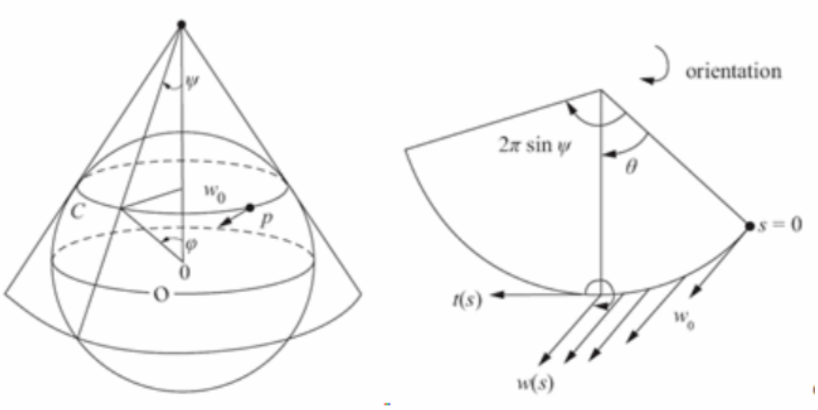
微分幾何的一個重大發展是 1917 年 Levi-Civita 的平行性(parallelism)，共變微分可以決定測地線與高斯曲率。
平行性與共變微分相互決定([微分幾何講稿]p.124~135)
http://web.math.sinica.edu.tw/math_media/d443/44301.pdf

§ 傅科擺 https://youtu.be/5YwF58NrKao
計算部分看[RG4102-2S^2] (g)

若傅科擺位於北緯 λ 度($\lambda = \frac{\pi}{2} - \theta_0$)， 則擺動方向每 24 小時轉動 $2\pi \sin\lambda$

參考 [大域微分幾何]p.156~162

3. 傅科擺(小說)
4. P.do Carmo p.431 有古典的解法



5. [大人的理科教室] p.52 有詳細的圖解。傅科擺看起來像是因為受到外力而轉動的，這個外力則是科氏力(Coriolis force)。
6. 傅科擺的物理 https://highscope.ch.ntu.edu.tw/wordpress/?p=46671
7. 傅科擺 https://highscope.ch.ntu.edu.tw/wordpress/?p=19258
8. Foucault 與 Levi-Civita 張海潮
https://web.math.sinica.edu.tw/mathmedia/media18.jsp?voln=234
9. From the geometry of Foucault pendulum to the topology of planetary waves
Pierre Delplace Antoine Venaille [ResearchGate]