§ Comparison theorem

曲率來刻劃完備黎曼流形的拓撲性質是近代微分幾何發展的路徑之一。

首先是用曲率做一些幾何性質的比較，即對兩黎曼流形的曲率關係比較相應的微分幾何與拓撲性質，然後研究標準空間(例如常曲率流形)的性質，這方面的結果稱為比較定理。

In Riemannian geometry，the comparison results in terms of sectional curvature of Rauch，Toponogov，Morse-Schoenberg and others are basic tools used to prove resucherlts such as sphere theorem，the Bonnet-Myers theorem，and the maximal diameter theorem of Toponogov。

1. Harry E. Rauch 1925-1979 A Contribution to Diffreential Geometry in the Large
   Lectures on the Ricci flow Peter Topping [Using Rauch comparison theorem to get an estimation of two metric]
2. Victor Andreevich Toponogov 1903-2004 [Toponogov's theorem and Applications]
   [International conference on Geometry in the Large] [A relative Toponogov comparison theorem]

More recently，comparison theorems in terms of the Ricci curvature such as Bishop-Gromov volume comparison theorem have played an important role leading to such results as the Chern maximal diameter theorem。

Hermann Karcher :Riemannian comparison constructions

1. Scalar curvature
2. Ricci curvature Bishop-Gromov relative volume comparison theorem
3. Sectional curvature Rauch Toponogov Morse-Schoenberg theorem

1. Hadamard-Cartan theorem
2. Bonnet-Meyers theorem
3. Morse-Schonberg comparison theorem
4. Rauch comparison theorem
5. Bishop-Gromov relative volume comparison theorem

截曲率(Sectional curvature)

$K(\pi):=<R(e_1,e_2)e_2,e_1>$ where $\{e_1,e_2\}$ is an orthonormal basis of π

Prove $K(\pi)=\dfrac{<R(X,Y)Y,X>}{|X|^2|Y|^2-<X,Y>^2}$

One important class of results in Riemannian geometry are the "gap type" rigidity theorems. Let us mention a few examples of model gap results. We do not try to state these results under the weakest hypotheses. Let (M,g) be a complete connected Riemannian manifold, isometric to Euclidean $\mathbb{R}^n$ in the complement of a compact set with $n \geq 3$.

- **Scalar curvature:** Assume that (M,g) is spin and has non–negative scalar curvature. Then (M,g) is isometric to Euclidean $\mathbb{R}^n$. This follows from the Witten argument for the positive mass theorem.
- **Ricci curvature:** Assume that (M,g) has non–negative Ricci curvature. Then the conclusion follows from the Bishop–Gromov volume comparison theorem.
- **Sectional curvature:** If (M,g) has either non–negative or non–positive sectional curvature, then theorems of Greene and Wu [16] (for non-negative sectional curvature) and Kasue and Sugahara [24] (for non-positive sectional curvature) imply (M,g) is isometric to $\mathbb{R}^n$.

Hadamard-Cartan theorem：

Riemann 流形 M 是完備的，且任意截曲率 $K_M \leq 0$ ，則

1. M 中任兩點都不互相共軛，且因此 $Exp_p : T_pM \to M$ 為 local diffeomorphism。
2. 若 M 又單連通則 $M \overset{diffeo}{\approx} R^n$

The Hadamard-Cartan Theorem 1. *Let M be complete.*
(1) Let $\gamma : [0, \beta] \to M$ be a unit speed geodesic in M such that

$$\mathcal{K} \leq 0 \quad \text{for all sectional curvatures along } \gamma\Big|_{[0,\beta]}.$$

Then $\gamma\Big|_{[0,\beta]}$ contains no point conjugate to $\gamma(0)$ along γ.
(2) Therefore, if M is complete and all sectional curvatures are nonpositive, then M has no conjugate points.

The Hadamard-Cartan Theorem 2. *If M is complete, and all of its sectional curvatures are nonpositive, then for any $p \in M$, $\exp_p : T_pM \to M$ is of maximal rank.*

Theorem 3. *Suppose $\widetilde{M}$ and M are connected Riemannian manifolds, with $\widetilde{M}$ complete, and $\pi : \widetilde{M} \to M$ is a local isometry. Then M is complete and π is a covering map.*

The Hadamard-Cartan Theorem 4. *If M is complete, and all of its sectional curvatures are nonpositive, then for any $p \in M$, $\exp_p : T_pM \to M$ is a covering.*

Theorem 5 (The Cartan-Hadamard Theorem). *If M is a complete, connected manifold all of whose sectional curvatures are nonpositive, then for any point $p \in M$, $\exp_p : T_pM \to M$ is a covering map. In particular, the universal covering space of M is diffeomorphic to $\mathbb{R}^n$.*
- *If M is simply connected, then M itself is diffeomorphic to $\mathbb{R}^n$.*

- Because of this theorem, a complete, simply-connected Riemannian manifold with nonpositive sectional curvature is called a **Cartan-Hadamard manifold**.
- An immediate consequence of the Cartan-Hadamard theorem is that there are stringent topological restrictions on which manifolds can carry metrics of nonpositive sectional curvature.

Example. (1) If M is a product of compact manifolds $M_1 \times M_2$ where either M_1 or M_2 is simply connected (such as, for example, $S^1 \times S^2$), then any metric on M must have positive sectional curvature somewhere.
(2) Any manifold whose universal cover is contractible is **aspherical**, which means that the higher homotopy groups $\pi_k(M)$ vanish for $k > 1$, so many manifolds cannot admit metrics of nonpositive curvature.

Proof. The assumption of nonpositive curvature guarantees that p has no conjugate points along any geodesic.
— Therefore, $\exp_p$ is a local diffeomorphism on all of T_pM.
— Let $\tilde{g}$ be the (variable-coeffient) 2-tensor field $\exp_p^* g$ defined on T_pM.
— Because $\exp_p^*$ is everywhere nonsingular, $\tilde{g}$ is a Riemannian metric, and $\exp_p : (T_pM, \tilde{g}) \to (M, g)$ is a local isometry.
— It then follows from Theorem 3 that $\exp_p$ is a covering map.
— The remaining statement of the theorem follow immediately from uniqueness of the universal covering space. □

Bonnet-Meyers theorem：

M 是完備、連通的曲面，且 $Ric_M \geq (n-1)K_0 g$ ， K_0 是大於 0 的常數。則 M 必為緊緻，且其直徑 $diamM \leq \frac{\pi}{\sqrt{K_0}}$ ， $diam(M) =: \sup\{d(p,q) | p,q \in M\}$

古典的曲面論，Bonnet 定理：

一個完備的曲面，若其高斯曲率 $K \geq K_0 > 0$ K，則 M 必為緊緻而其直徑 $diamM \leq \frac{\pi}{\sqrt{K_0}}$ 。

Bonnet's Theorem 1. *Let M be a* **complete**, *connected Riemannian manifold all of whose sectional curvatures are bounded below by a positive constant $1/R^2$. Then M is compact with diameter less than or equal to πR.*

Meyer Theorem 1. *Suppose M is a complete, connected Riemannian n-manifold whose Ricci tensor satisfies the following inequality for all $V \in TM$:*

$$Ric(V,V) \geq \frac{n-1}{R^2}|V|^2;$$

Then M is compact and with diameter less than or equal to πR.

- One of the most important applications of Meyer's theorem is to **Einstein metrics**.

— If g is a complete Einstein metric with positive scalar curvature, then Ric$=\frac{1}{n}Sg$ satisfies the hypotheses of the theorem.
— It follows that complete, noncompact Einstein manifolds must have nonpositive scalar curvature.

- On the other hand, it is possible for complete, noncompact manifolds to have strictly positive Ricci or even sectional curvature, as long as it gets arbitrarily close to zero.

Proposition 1. *If $\psi : \widetilde{M} \to M$ is a Riemannian covering, then M is complete iff $\widetilde{M}$ is complete.*

Theorem 2 (Myers (1941)). *If M satisfies the hypothesis of the Bonnet-Myers Theorem; i.e. M is complete and the Ricci curvature of M is bounded from below by a positive constant, then not only M is compact, but also that any cover of M, $\widetilde{M}$ is compact.*

Proof. Let $\pi : \widetilde{M} \to M$ denote a covering space of M, with the metric $\widetilde{g} = \pi^* g$.
$\widetilde{M}$ is complete by Proposition 1,
and $\widetilde{g}$ and g have sectional curvature bounded below by the same constant,
so $\widetilde{M}$ is compact by the argument above. □

Bonnet's Theorem 2. *Let M be a* **complete**, *connected Riemannian manifold all of whose sectional curvatures are bounded below by a positive constant $1/R^2$. Then M is compact, with a finite fundamental group, and with diameter less than or equal to πR.*

Proof. **Claim: $\pi_1(M)$ is finite.**
— Let $\pi : \widetilde{M} \to M$ denote the universal covering space of M, with the metric $\widetilde{g} = \pi^* g$.
— $\widetilde{M}$ is complete by Proposition 1, and $\widetilde{g}$ also has sectional curvature bounded below by $1/R^2$, so $\widetilde{M}$ is compact by the argument above.
— There is one-to-one correspondence between $\pi_1(M)$ and the inverse image $\pi^{-1}(p)$ of any point $p \in M$.
— If $\pi_1(M)$ were infinite, therefore, $\pi^{-1}(p)$ would be an infinite discrete set in $\widetilde{M}$, contradicting the compactness of $\widetilde{M}$. □

Meyer Theorem. *Suppose M is a complete, connected Riemannian n-manifold whose Ricci tensor satisfies the following inequality for all $V \in TM$:*

$$Ric(V, V) \geq \frac{n-1}{R^2}|V|^2;$$

Then M is compact with diameter less than or equal to πR and with a finite fundamental group,.

Morse-Schonberg comparison theorem

Rauch comparision theorem

Let (M, g) and $(\widetilde{M}, \tilde{g})$ be Riemannian manifolds of dimension m. Let

$$\gamma : [0, a] \to M \quad \text{and} \quad \tilde{\gamma} : [0, a] \to \widetilde{M}$$

be normal geodesics with

$$\gamma(0) = p, \tilde{\gamma}(0) = \tilde{p}.$$

For each $t \in [0, a]$, we denote

$$K^-(t) = \min\{K(\Pi_{\gamma(t)}) \mid \dot{\gamma}(t) \in \Pi_{\gamma(t)}\},$$
$$\widetilde{K}^+(t) = \max\{\widetilde{K}(\widetilde{\Pi}_{\tilde{\gamma}(t)}) \mid \dot{\tilde{\gamma}}(t) \in \widetilde{\Pi}_{\tilde{\gamma}(t)}\}.$$

Theorem 1.1 (Rauch comparison theorem). *Let $X, \widetilde{X}$ be Jacobi fields along $\gamma, \tilde{\gamma}$ respectively, such that*

①$X(0) = 0, \widetilde{X}(0) = 0,$ ②$|\nabla_{\dot{\gamma}(0)} X| = |\widetilde{\nabla}_{\dot{\tilde{\gamma}}(0)} \widetilde{X}|,$ ③$\langle \dot{\gamma}(0), \nabla_{\dot{\gamma}(0)} X \rangle = \langle \dot{\tilde{\gamma}}(0), \widetilde{\nabla}_{\dot{\tilde{\gamma}}(0)} \widetilde{X} \rangle.$

Assume further that

ⓘγ *has no conjugate points on* $[0, a]$, ⓘⓘ$\widetilde{K}^+(t) \leq K^-(t)$ *holds for all* $t \in [0, a]$.

Then $\tilde{\gamma}$ *has no conjugate points on* $[0, a]$, *and for all* $t \in [0, a]$,

$$|X(t)| \leq |\widetilde{X}(t)|.$$

Theorem 1.3 (Rauch comparison theorem, Second form). *Suppose* ⓘ *and* ⓘⓘ *holds. Denote* $p = \gamma(0)$ *and* $\tilde{p} = \tilde{\gamma}(0)$, *and suppose* $X_p \in T_p M, \widetilde{X}_{\tilde{p}} \in T_{\tilde{p}} \widetilde{M}$ *satisfies*

$$\langle X_p, \dot{\gamma}(0) \rangle = \langle \widetilde{X}_{\tilde{p}}, \dot{\tilde{\gamma}}(0) \rangle, \quad |X_p| = |\widetilde{X}_{\tilde{p}}|.$$

Then $|(d \exp_p)_{t\dot{\gamma}(0)} X_p| \leq |(d \exp_{\tilde{p}})_{t\dot{\tilde{\gamma}}(0)} \widetilde{X}_p|$

一些應用暫略。

Definition 1.30. We define a function sn_k as follows:

$$\text{sn}_k(r) = \begin{cases} r & \text{if} \quad k = 0, \\ \frac{1}{\sqrt{k}} \sinh(\sqrt{k} r) & \text{if} \quad k > 0. \end{cases}$$

The function $\text{sn}_k(r)$ is the solution to the equation

$$\varphi'' - k\varphi = 0,$$
$$\varphi(0) = 0,$$
$$\varphi'(0) = 1.$$

We define $\text{ct}_k(r) = \text{sn}_k'(r) / \sqrt{k} \text{sn}_k(r)$.

Theorem 1.31. *(Sectional Curvature Comparison) Fix* $k \geq 0$. *Let* (M, g) *be a Riemannian manifold with the property that* $-k \leq K(P)$ *for every 2-plane* P *in* TM. *Fix a minimizing geodesic* $\gamma \colon [0, r_0) \to M$ *parameterized at unit speed with* $\gamma(0) = p$. *Impose Gaussian polar coordinates* $(r, \theta^1, \ldots, \theta^{n-1})$ *on a neighborhood of* γ *so that* $g = dr^2 + g_{ij} \theta^i \otimes \theta^j$. *Then for all* $0 < r < r_0$ *we have*

$$(g_{ij}(r, \theta))_{1 \leq i,j \leq n-1} \leq \text{sn}_k^2(r),$$

and the shape operator associated to the distance function from p, f, *satisfies*

$$(S_{ij}(r, \theta))_{1 \leq i,j \leq n-1} \leq \sqrt{k} \text{ct}_k(r).$$

There is also an analogous result for a positive upper bound to the sectional curvature, but in fact all we shall need is the local diffeomorphism property of the exponential mapping.

LEMMA 1.32. *Fix $K \geq 0$. If $|\mathrm{Rm}(x)| \leq K$ for all $x \in B(p, \pi/\sqrt{K})$, then $\exp_p$ is a local diffeomorphism from the ball $B(0, \pi/\sqrt{K})$ in T_pM to the ball $B(p, \pi/\sqrt{K})$ in M.*

There is a crucial comparison result for volume which involves the Ricci curvature.

THEOREM 1.33. **(Ricci curvature comparison)** *Fix $k \geq 0$. Assume that (M, g) satisfies $\mathrm{Ric} \geq -(n-1)k$. Let $\gamma\colon [0, r_0) \to M$ be a minimal geodesic of unit speed. Then for any $r < r_0$ at $\gamma(r)$ we have*

$$\sqrt{\det g(r, \theta)} \leq \mathrm{sn}_k^{n-1}(r)$$

and

$$\mathrm{Tr}(S)(r, \theta) \leq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}.$$

THEOREM 1.34. *(Relative Volume Comparison, Bishop-Gromov 1964-1980) Suppose (M, g) is a Riemannian manifold. Fix a point $p \in M$, and suppose that $B(p, R)$ has compact closure in M. Suppose that for some $k \geq 0$ we have $\mathrm{Ric} \geq -(n-1)k$ on $B(p, R)$. Recall that H_k^n is the simply connected, complete manifold of constant curvature $-k$ and $q_k \in H_k^n$ is a point. Then*

$$\frac{\mathrm{Vol}\, B(p, r)}{\mathrm{Vol}\, B_{H_k^n} B(q_k, r)}$$

is a non-increasing function of r for $r < R$, whose limit as $r \to 0$ is 1. In particular, if the Ricci curvature of (M, g) is ≥ 0 on $B(p, R)$, then $\mathrm{Vol}\, B(p, r)/r^n$ is a non-increasing function of r for $r < R$.