§ Ricci Soliton

(M,g) is a Riemannian manifold，X is a smooth vector field satisfy

(2.1) $Ric(g)+\frac{1}{2}L_X g=\frac{\lambda}{2}g$ tracing it we get：

(2.2) $R+divX=\frac{n\lambda}{2}$ where $divX=tr(\nabla X)=\sum_i \nabla_i X^i$

The most important class of Ricci solitons is those for which $X=\nabla f$ for some smooth function *f* on *M*。

For these so-called **gradient Ricci solitons**，equation (2.1) simplifies to (2.3)

$$Ric(g)+\nabla^2 f=\frac{\lambda}{2}g$$

The groups R+ of positive real numbers and Diff(*Mn*) of diffeomorphisms act naturally by dilation via $\alpha \cdot g = \alpha g$ and pull back via $\varphi \cdot g = \varphi^* g$, respectively, on the space Met(*Mn*) of Riemannian metrics on *Mn*. Via the scaling and diffeomorphism invariances

(2.4) Ric(αg) = Ric(g), Ric($\varphi^* g$) = φ^*Ric(g)

of the Ricci tensor, they act on Ricci solitons (*Mn, g, X,* λ) as follows：

(1) (Metric scaling) If $\alpha \in R^+$，then $(\mathrm{M},\alpha g,\alpha^{-1}X,\alpha^{-1}\lambda)$ is a Ricci soliton。

(2) (Diffeomorphism invariance) If $\phi: N \to M$ is a diffeomorphism，then $(N,\phi^* g,\phi^* X,\lambda)$ is a Ricci soliton。

1. Gradient 梯度
2. $L_{\nabla f} g = 2\nabla^2 f$，here ∇^2 denotes the Hessian，i.e. the second covariant derivative。This acts on tensors，and when acting on a function f，$\nabla^2 f = \nabla df$。
3. If K is a Killing vector field，then $(M,g,X+K,\lambda)$ is a Ricci soliton。