The Morse Index Theorem

H.C. Marston Morse 1892-1977

It relates the number of **conjugate points** along a geodesic in a Riemannian manifold to the **index**of a certain quadratic form associated with the geodesic。The theorem is named after Marston Morse，who made significant contributions to the calculus of variations and critical point theory。

Key concepts：

1. **Geodesic**：A curve in a Riemannian manifold that locally minimizes length (or energy) between two points。It satisfies the geodesic equation derived from the metric。
2. **Conjugate Points**：Two points*p*and*q*along a geodesic are called conjugate if there exists a non-trivial Jacobi field (a solution to the Jacobi equation) that vanishes at both*p*and*q*。Conjugate points indicate a kind of "focusing" of nearby geodesics。
3. **Index Form**：The index form is a quadratic form defined on the space of vector fields along a geodesic。It arises from the second variation of the length or energy functional。The index of this form counts the number of independent directions in which the geodesic can be deformed to decrease its length (or energy)。
4. **Morse Index**：The Morse index of a geodesic is the number of negative eigenvalues of the index form。Intuitively，it measures the number of independent directions in which the geodesic is "unstable" (i.e not a local minimum of the length or energy functional)。

The Morse Index theorem：

The **Morse index** of a geodesic segment (the number of **negative eigenvalues** of the index form) is equal to the **number of conjugate points** along the geodesic，counted with multiplicity。

In other words，the Morse index of a geodesic is the total number of conjugate points between the endpoints of the geodesic，where each conjugate point contributes to the index based on its multiplicity。

Intuition：

1. Conjugate points represent "obstructions(障礙物)" to minimizing properties of the geodesic。Each conjugate point corresponds to a direction in which the geodesic fails to be a local minimum。
2. The Morse index counts how many such obstructions exist along the geodesic。

Applications：

1. The Morse Index Theorem is used in the study of geodesics，minimal surfaces，and Hamiltonian systems。
2. It plays a key role in understanding the stability of geodesics and the structure of the space of paths in a Riemannian manifold。
3. It is also important in global analysis， particularly in the study of the topology of manifolds via Morse theory。

Example

On a sphere，consider a geodesic (a great circle) connecting the north pole to the south pole。 The south pole is a conjugate point to the north pole because all great circles focus at the south pole。The Morse index of this geodesic is 1，corresponding to the single conjugate point (the south pole)。

Definition of an index form：

Let $\gamma : [a, b] \to M$ be a geodesic in a Riemannian manifold (M, g). The **index form** is a bilinear form defined on the space of vector fields along γ that vanish at the endpoints. It is given by:

$$I(V, W) = \int_a^b \left(g(\nabla_{\dot{\gamma}} V, \nabla_{\dot{\gamma}} W) - g(R(\dot{\gamma}, V)\dot{\gamma}, W) \right) dt,$$

The index form arises from the **second variation of the energy functional** $E(\gamma)=\frac{1}{2}\int_a^b g(\dot{\gamma},\dot{\gamma})dt$ 。If γ is a geodesic，the first variation of E vanishes， and the second variation is given by the index form。Specifically，for a variation γ_s of γ with variation vector field V，the second derivative of the energy is： $\frac{d^2}{ds^2}E(\gamma_s)\big|_{s=0}=I(V,V)$

Thus，the index form measures how the energy changes to second order when the geodesic is perturbed in the direction of V。

Role in Stability Analysis:

- If $I(V, V) > 0$ for all non-zero V, the geodesic is a **local minimum** of the energy functional.
- If $I(V, V) < 0$ for some V, the geodesic is **unstable** (not a local minimum).
- The **Morse index** of the geodesic is the number of independent directions V for which $I(V, V) < 0$. This corresponds to the number of negative eigenvalues of the index form.

Connetion to conjugate points：

The index form is closely related to the existence of **conjugate points** along the geodesic。A conjugate point occurs when there is a non-trivial Jacobi field (a solution to the Jacobi equation) that vanishes at two points along the geodesic。 The presence of conjugate points affects the sign of the index form and thus the stability of the geodesic。

Summary：

The index form is a quadratic form that encodes information about the second variation of a functional (e.g., energy or length) along a geodesic. It is used to:

1. Determine the stability of geodesics
2. Count the number of conjugate points (via the Morse Index Theorem)
3. Study the minimizing properties of geodesics in Riemannian geometry

§ Conjugate points on a cylinder

Consider a**cylinder** $M = S^1 \times R$ with the standard flat metric。Geodesics on the cylinder are of two types：

1. **Vertical lines**：These are straight lines parallel to the R-axis。
2. **Helices**：These are curves that wrap around the cylinder at a constant angle。

We will focus on the **helical geodesics** to study conjugate points。

A helical geodesic can be written as：$\gamma(t)=(\cos t,\sin t,ct)$,
where c is a constant determining the "slope" of the helix。

Jacobi Fields and Conjugate Points:

To find conjugate points, we need to study **Jacobi fields** along γ。A Jacobi field $J(t)$ is a vector field along γ that satisfies the **Jacobi equation**：

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} J + R(\dot{\gamma}, \mathrm{J})\dot{\gamma} = 0$$

where R is the Riemann curvature tensor. For a flat cylinder，R=0, so the Jacobi equation simplifies to：$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} J = 0$

This is just the equation for a **parallel vector field** along γ。

Solving the Jacobi fields：

On the cylinder, the general solution to the Jacobi equation is:

$$J(t) = (A\cos t + B\sin t, -A\sin t + B\cos t, Dt + E),$$

where A, B, D, E are constants.

To find conjugate points, we look for non-trivial Jacobi fields $J(t)$ that vanish at two distinct points $t = t_1$ and $t = t_2$.

Conjugate Points on the Cylinder:

1. **Vertical Component**: The z-component of $J(t)$ is $Dt + E$. For $J(t)$ to vanish at two distinct points, we must have $D = 0$ and $E = 0$. This means the vertical component of $J(t)$ is trivial.
2. **Horizontal Component**: The horizontal component of $J(t)$ is:

   $$(A\cos t + B\sin t, -A\sin t + B\cos t).$$

   For $J(t)$ to vanish at two distinct points, we must have $A = 0$ and $B = 0$. This means the horizontal component of $J(t)$ is also trivial.

Conclusion：

On a flat cylinder，**there are no conjugate points** along any geodesic。This is because the only Jacobi fields that vanish at two distinct points are the trivial ones (i.e.,$J(t)=0$for all t)。

§ a torus

Consider a flat torus $T^2 = S^1 \times S^1$

Geodesics on the torus fall into two categories:

1. **Closed geodesics**: These are geodesics that wrap around the torus periodically. They correspond to lines in $\mathbb{R}^2$ with rational slope.
2. **Dense geodesics**: These are geodesics that fill the torus densely. They correspond to lines in $\mathbb{R}^2$ with irrational slope.

Conjugate Points on the Torus:

To study conjugate points, we again analyze **Jacobi fields** along geodesics. On a flat torus, the Riemann curvature tensor $R = 0$, so the Jacobi equation simplifies to:

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J = 0.$$

This means Jacobi fields are linear in the parameter t, just like on the cylinder.

Case 1: Closed Geodesics (Rational Slope)

Consider a closed geodesic γ with rational slope $\frac{p}{q}$, where p and q are integers. This geodesic wraps around the torus p times in one direction and q times in the other before closing.

- **Jacobi Fields**: The general solution to the Jacobi equation is:

$$J(t) = (At + B, Ct + D),$$

 where A, B, C, D are constants.

- **Conjugate Points**: For $J(t)$ to vanish at two distinct points $t = t_1$ and $t = t_2$, we must have $A = C = 0$ and $B = D = 0$. This means the only Jacobi field that vanishes at two distinct points is the trivial one ($J(t) = 0$).

Thus, **closed geodesics on the flat torus do not have conjugate points**.

Case 2: Dense Geodesics (Irrational Slope)

Consider a dense geodesic γ with irrational slope. This geodesic never closes and fills the torus densely.

- **Jacobi Fields**: The Jacobi fields are again linear:

$$J(t) = (At + B, Ct + D).$$

- **Conjugate Points**: As in the closed case, the only Jacobi field that vanishes at two distinct points is the trivial one. Thus, **dense geodesics on the flat torus also do not have conjugate points**.

On a **flat torus**, **there are no conjugate points** along any geodesic. This is because:

1. The torus is flat ($R = 0$), so the Jacobi equation reduces to $\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} J = 0$, and Jacobi fields are linear.
2. The only Jacobi field that vanishes at two distinct points is the trivial one ($J(t) = 0$).

- **Sphere**: Geodesics are great circles, and conjugate points occur at antipodal points. The Morse index of a geodesic segment longer than half the circumference is at least 1.
- **Cylinder**: Geodesics are helices or vertical lines, and there are no conjugate points.
- **Torus**: Geodesics are straight lines (in the universal cover), and there are no conjugate points.