§ Ricci Scalar for the BH spacetime

We consider a charged BH surrounded quintessence with the equation of state parameter $\varepsilon = \frac{p\phi}{\sigma\phi}$，$-1 < \varepsilon < -\frac{1}{3}$，and α is the normalization constant。

Quintessence 第五元素 a hypothetical form of dark energy ，a scalar field

The metric of a charged BH reads：

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Where $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r} - \frac{\alpha}{r^{3\varepsilon+1}}$

Where M and Q represent the mass and charge of the BH respectively。

The Ricci scalar for the BH spacetime is $R = \frac{3\alpha\varepsilon(1-3\varepsilon)}{r^{3\varepsilon+3}}$

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_m \Gamma^m_{jk}\Gamma^l_{im} - \sum_m \Gamma^m_{ik}\Gamma^l_{jm}$$

$$d\omega^i = \omega^j \wedge \omega^i_j$$

$$\Omega^j_i = d\omega^j_i - \omega^k_i \wedge \omega^j_k$$

$$R^j_{iji} = \Omega^j_i(E_i, E_j) \text{，} R_{ij} = R^k_{kij}$$

$$R = \sum_{i,j} g^{ij} R_{ij}$$

$L = f(r)\dot{t}^2 - \frac{1}{f(r)}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2$ ，其中•表示對 s 微分，'表示對 r 微分

例如 the Euler-Lagrange equation for θ is

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{ds}(-2r^2\dot{\theta}) + 2r^2\sin\theta\cos\theta\dot{\phi}^2 = 0$$

$-4r\dot{r}\dot{\theta} - 2r^2\ddot{\theta} + 2r^2\sin\theta\cos\theta\dot{\phi}^2 = 0$ 即

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0$$

由此可知 $\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{r}, \Gamma^3_{44} = -\sin\theta\cos\theta$

同理 the Euler-Larange equation for ϕ is $\frac{d}{ds}(\frac{\partial L}{\partial \dot{\phi}})-\frac{\partial L}{\partial \phi}=0$ 得到

$\ddot{\phi}+\frac{2}{r}\dot{r}\dot{\phi}+2\cot\theta\dot{\theta}\dot{\phi}=0$，所以 $\Gamma^4_{24}=\Gamma^4_{42}=\frac{1}{r}$ ，$\Gamma^4_{34}=\Gamma^4_{43}=\cot\theta$

The geodesic equations for the metric are given by

$$\ddot{t}+\frac{f'(r)}{f(r)}\dot{r}\dot{t}=0 \qquad (1)$$

$$\ddot{r}+[\frac{f'(r)\dot{t}^2+f^{-1}(r)'\dot{r}^2-2r\dot{\theta}^2-2r\sin^2\theta\dot{\phi}^2}{2f^{-1}(r)}]=0 \qquad (2)$$

$$\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\cos\theta\sin\theta\dot{\phi}^2=0 \qquad (3)$$

$$\ddot{\phi}+\frac{2}{r}\dot{r}\dot{\phi}+2\cot\theta\dot{\theta}\dot{\phi}=0 \qquad (4)$$

得到 Γ^k_{ij}

$$\Gamma^1_{12}=\frac{1}{2}\times\frac{1}{f(r)}\times\frac{\partial f}{\partial r}=\Gamma^1_{21}$$

$$\Gamma^2_{11}=\frac{1}{2}f(r)\frac{\partial f(r)}{\partial r}\text{ ，}\Gamma^2_{22}=\frac{1}{2}f(r)\times f^{-1}(r)'\text{ ，}\Gamma^2_{33}=-rf(r)\text{ ，}$$

$$\Gamma^2_{44}=\frac{1}{2}(-f(r))(2r\sin^2\theta)$$

$$\Gamma^3_{23}=\Gamma^3_{32}=\frac{1}{r}\text{ ，}\Gamma^3_{44}=-\sin\theta\cos\theta$$

$$\Gamma^4_{24}=\Gamma^4_{42}=\frac{1}{r}\text{ ，}\Gamma^4_{34}=\Gamma^4_{43}=\cot\theta$$

換句話說 可以由定義直接得到 Christoffel symbols。
或者由 E-L 方程得到 geodesic 方程，再得到 Christoffel symbols。
那麼 我們該用 Christoffel symbols，還是用 Cartan structure 方程算 Ricci scalar？

Cartan structure equations

$$d\omega^i=\sum_j \omega^j\wedge\omega^i_j$$

$$\Omega^j_i=d\omega^j_i-\sum_k \omega^k_i\wedge\omega^j_k\text{ ，}R^j_{iji}=\Omega^j_i(E_i,E_j)\text{ ，}R_{ij}=R^k_{kij}\text{ ，}R=\sum_{i,j}g^{ij}R_{ij}$$