§

Riemann tensor $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Torsion $\mathrm{T(X,Y)} = \nabla_{\mathrm{X}} \mathrm{Y} - \nabla_Y X - [X,Y]$

$R\ {}_{\sigma\mu\nu}^{\rho} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}$

$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda_{\sigma\mu\nu}$

$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$, $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$, $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$, $R_{\rho[\sigma\mu\nu]} = 0$

Bianchi Identity :

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_\beta \partial_\mu g_{\nu\alpha} + \partial_\alpha \partial_\nu g_{\beta\mu} - \partial_\beta \partial_\nu g_{\alpha\mu} - \partial_\alpha \partial_\mu g_{\beta\nu})$$

$R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0$ L.Bianchi 1902

Where $R^h_{ijk,l}$:=covariant derivative of R^h_{ijk} w.r.t. the $l-th$ coordinate。

M is a manifold with symmetric connection $\nabla\,(\nabla_X Y - \nabla_Y X = [X,Y])$，then

$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

Show that the sum of cyclic permutations of the last three indices of the curvature tensor vanishes, i.e.

$$R_{\kappa\lambda\mu\nu} + R_{\kappa\mu\nu\lambda} + R_{\kappa\nu\lambda\mu} = 0\,, \qquad 1^{\text{st}}\ \text{Bianchi identity}\,. \tag{6}$$

Make use of locally inertial coordinates once more to prove

$$\nabla_{[\mu} R_{\kappa\lambda]\rho\sigma} = 0\,, \qquad 2^{\text{nd}}\ \text{Bianchi identity}\,. \tag{7}$$

By contracting indices of the second Bianchi identity (7) twice, show that

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R\,.$$

Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$

Scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$

Torsion tensor $T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 2\Gamma^{\lambda}_{[\mu\nu]}$ then $[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = R^{\rho}_{\sigma\mu\nu}V^{\sigma} - T^{\lambda}_{\mu\nu}\nabla_{\lambda}\mathrm{V}^{\rho}$

$\nabla_{\mu}\nabla_{\nu}V^{\rho} = \partial_{\mu}(\nabla_{\nu}V^{\rho}) - \Gamma^{\lambda}_{\mu\nu}\nabla_{\lambda}V^{\rho} + \Gamma^{\rho}_{\mu\sigma}\nabla_{\nu}V^{\sigma}$ =....