§ Induced metric

Two Riemann manifolds (M_i, g_i) are said to be isometric if there exists a

diffeomorphim $\phi: M_1 \to M_2$ such that $\phi^* g_2 = g_1$

$\varphi: M \to N$ is an immersion。(N, g) is a Riemannian manifold，then $\varphi^* g$ is a Riemannian metric in M induced by φ

§ Example

$\phi: S^2 \to R^3$, $\phi:(0,\pi)\times(0,2\pi) \to R^3$

Then $\phi^* g = d\theta^2 + \sin^2\theta d\varphi^2$ is the induced metric on S^2

(R^3, g) $g = (dx)^2 + (dy)^2 + (dz)^2$

$(S^2, \tilde{g})$ ， $\tilde{g} = \varphi^* g$

$$\frac{\partial}{\partial\theta} = \frac{\partial\phi}{\partial\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\frac{\partial}{\partial\varphi} = \frac{\partial\phi}{\partial\varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

$$\tilde{g}_{11} = g_{\theta\theta} = <\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}> = 1 ,$$

$$\tilde{g}_{12} = \tilde{g}_{21} = g_{\theta\varphi} = g_{\varphi\theta} = <\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}> = 0 , \ \tilde{g}_{22} = g_{\varphi\varphi} = <\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}> = \sin^2\theta$$

$$\therefore \varphi^* g = d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi$$

Or ，on S^2 $x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta$

$$dx = \cos\theta\cos\varphi dx - \sin\theta\sin\varphi d\varphi$$
$$dy = \cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi$$
$$dz = -\sin\theta d\theta$$

Then $dx^2 + dy^2 + dz^2 = \ldots = d\theta^2 + \sin^2\theta d\varphi^2$

$(S^2, h) \xrightarrow{\phi} (R^3, g)$ ， $g = dx^2 + dy^2 + dz^2, h = d\theta^2 + \sin^2\theta d\varphi^2$

$\phi^* g = d\theta^2 + \sin^2\theta d\varphi^2 = h$

§ Example

$(R^3, g), g = (dx)^2 + (dy)^2 + (dz)^2$ ，$\varphi : S^2 \to R^3$ 求 $\varphi^* g$

$\varphi(x,y) = (x, y, \sqrt{1-x^2-y^2})$

$\frac{\partial\varphi}{\partial x} = (1, 0, \frac{-x}{\sqrt{1-x^2-y^2}})$ ，$\frac{\partial\varphi}{\partial y} = (0, 1, \frac{-y}{\sqrt{1-x^2-y^2}})$

$g_{11} = < \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial x} > = \frac{1-y^2}{1-x^2-y^2}$ ，$g_{12} = g_{21} = < \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y} > = \frac{2xy}{1-x^2-y^2}$ ，

$g_{22} = < \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial y} > = \frac{1-x^2}{1-x^2-y^2}$

$So \ \ \varphi^* g = \frac{1-y^2}{1-x^2-y^2}(dx)^2 + \frac{2xy}{1-x^2-y^2} dxdy + \frac{1-x^2}{1-x^2-y^2}(dy)^2$

Then φ is called isometric embedding(同維度時才是 isometry(等距同構)。

§ Exercise

$C : \{(x,y,z) \in R^3 \mid x^2 + y^2 = 1\}$ is a cylinder

$\varphi(x,y,z) = (x, -y, -z) : C \to C$ then

(1) φ is an isometry

(2) $\varphi(p) = p$ ，p=?

(2) P(1,0,0) (-1,0,0)

§ 流形 M 上存在處處可微、恆正、二階對稱協變張量場 G，稱為度規(metric)張量場。

$g_{ij} = g_{ji}, \det(g_{ij}) \neq 0$

恆正改為非奇異(nonsingular)，則相應的流形 M 稱為廣義黎曼流形。

在 4-dim 的 Spacetime 上的流形是廣義的黎曼流形。

定義 $ds^2 = g_{ij} dx^i dx^j$ ，則弧長 $\Delta s = \int (g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt})^{\frac{1}{2}} dt$

利用 metric 可在流形 M 上每一點 p 的切空間 $T_p M$ 給出兩切向量的內積，定義向量的長度，兩向量的夾角。

利用聯絡結構可定義張量場的協變微分與平行運輸(parallel transport)，向量場依此聯絡平行運輸時保持向量長度不變及向量間夾角不變，即度規張量場 G 的協變微分為零， $\nabla G = 0$ ，取局部座標時可表為 $\nabla g = 0$

§ hyperbolic geometry

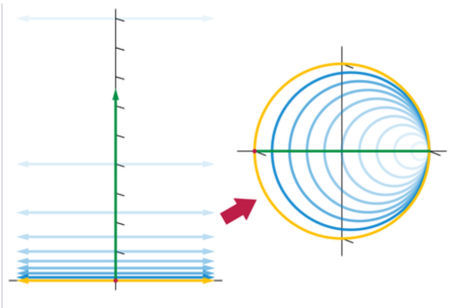
Let $H = \{(u,v) \in R^2 \,|\, v > 0\}$ denote the upper half plane。

Endowed with the metric $h = \dfrac{1}{4v^2}(du^2 + dv^2)$

The Poincare model of the hyperbolic plane is the Riemann manifold (D , g) where D is the unit open disk in the plane R^2 and the metric g is given by

$$g = \frac{1}{1 - x^2 - y^2}(dx^2 + dy^2)$$

Show that the Cayley transform $z = x + iy \to w = -i\dfrac{z+i}{z-i} = u + iv$ establishes an isometry $(D, g) \cong (H, h)$



Cayley transform of upper complex half-plane to unit disk

$\varphi(z) = \dfrac{z-i}{z+i} : H \to D$ is the Cayley transform，

which maps $\{\infty, 1, -1, i, 0\} \to \{1, -i, i, 0, -1\}$

$z = x \in R$ ，then $u + iv = \dfrac{x-i}{x+i} \Rightarrow u^2 + v^2 = 1$

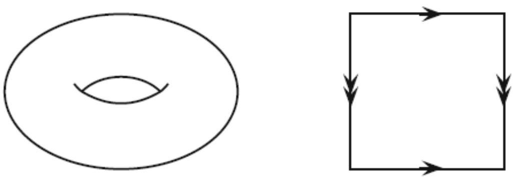
i.e. $x - axis \xrightarrow{\varphi} unit$ circle

Note　about Mobius transform

If $f(z) = \dfrac{az+b}{cz+d}$ then $f^{-1}(z) = \dfrac{-dz+b}{cz-a}$;

If $f(z) = -i \times \dfrac{z+i}{z-i}$ then $f^{-1}(z) = \dfrac{iz+1}{z+i} = i \times \dfrac{z-i}{z+i}$

§ A torus T^2 in R^3 inherits the Euclidean metric from R^3。



A torus is also $T^2 \cong R^2 / Z^2$ as a quotient space，it inherits a Riemannian metric from R^2。

Fig. 1.2. Two Riemannian metrics on the torus.

With these two Riemannian metrics，the torus becomes two distinct Riemannian manifolds。

Show that there is no isometry between these two Riemannian manifolds with the same underlying torus。

§ Metric connection

We say that a connection ∇ on a Remannian bundle E is compatible with the metric if for all $X \in \chi(M)$ and $s,t \in \Gamma(E)$

$$X < s,t > = < \nabla_X s,t > + < s,\nabla_X t >$$

Then ∇ is called metric connection。

「相容 $X \cdot < Y,Z > = < \nabla_X Y,Z > + < Y,\nabla_X Z >$

對稱 $\nabla_X Y - \nabla_Y X = [X,Y]$(called torsion free)

When ∇ compatible with metric and torsion free then ∇ is called Levi-Civita connection(Riemannian connection)。」