§ Metrics

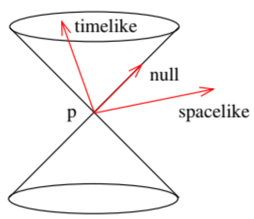
$g(X,Y):T_pM\times T_pM\to R$ is a (0,2) tensor，satisfies

1. g(X,Y)=g(Y,X)
2. $g(X,Y)\geq 0$
3. $g(X,X)=0\Leftrightarrow X=0$

With a choice of coordinates，then $g=g_{\mu\nu}dx^\mu dx^\nu$

§ Lorentzian manifolds

For example $\eta=-dx^0\otimes dx^0+dx^1\otimes dx^1+\ldots+dx^{n-1}\otimes dx^{n-1}$



The lightcone at a point p, with three different types of tangent vectors.

At any point p，a vector $X_p\in T_pM$ is

said to be timelike if $g(X_p,X_p)<0$

Null if $g(X_p,X_p)=0$

Spacelike if $g(X_p,X_p)>0$

A curve is called timelike if its tangent vector is everywhere timelike。In this case，the distance between two point p and q is $\tau=\int_a^b\sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}dt$ is called proper time。

1. Hyperbolic plane $ds^2=\frac{1}{y^2}(dx^2+dy^2)$
2. $I\times S^2$ $g=A^2(r)dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2$
3. S^3 $ds^2=d\psi^2+\sin^2\psi(d\theta^2+\sin^2\theta d\varphi^2)$
4. Cosmology $M=R\times\Sigma$，$g=-dt^2+a^2(t)(\frac{1}{1-kr^2}dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2)$

   Friedmann-Lemaitre-Robertson-Walker model of cosmology。
5. MT Wormhole $ds^2=-c^2dt^2+dl^2+(b_0^2+l^2)(d\theta^2+\sin^2\theta d\phi^2)$
6. Schwarzschild metric

$$ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2 \quad , \quad d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

This metric has two singularities：(1) when r=0 (2)when $r = r_s$ ，where $r_s = 2M$ is the Schwarzschild radius。

或者寫成 $ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\phi^2$ ，其中 $f(r) = 1-\frac{2GM}{r}$

張海潮先生的文章中寫成：

$$c^2 d\tau^2 = c^2(1-\frac{2GM}{rc^2})dt^2 - (1-\frac{2GM}{rc^2})^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2$$

其中 M 是太陽的質量，c 是慣性座標下真空中的光速。

7. Charged black hole

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Where $f(r) = 1-\frac{2M}{r} + \frac{Q^2}{r} - \frac{\alpha}{r^{3\varepsilon+1}}$

8. Kerr black hole

$$ds^2 = -\left(1-\frac{2GMr}{\rho^2}\right)\mathrm{d}t^2 - \frac{2GMar\sin^2\theta}{\rho^2}(\mathrm{d}t\,\mathrm{d}\phi + \mathrm{d}\phi\,\mathrm{d}t) + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\phi^2, \tag{6.70}$$

where

$$\Delta(r) = r^2 - 2GMr + a^2 \tag{6.71}$$

and

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta. \tag{6.72}$$

$$ds^2 = -[1-\frac{2mr}{r^2+a^2\cos^2\theta}](du + a\sin^2\theta d\phi)^2$$

$$+2(du + a\sin^2\theta d\phi)(dr + a\sin^2\theta d\phi) + (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2)$$

習作

1. Consider (R^2, g) to be the Riemannian manifold，with metric given by

$$g = (e^{-x} + y^2 e^x)dx^2 + xye^{-\frac{x}{2}}dxdy + 10(x^4 + y^4 + 5)dy^2$$

(a) Argue that this is a Riemannian metric

(b) Is this a complete manifold？Prove or give a reason why it would not be。

2. On R^3，consider the following metric

$$ds^2 = dx^2 + dy^2 + (dz + \sin z dx + \cos z dy)^2$$

(a) Calculate the Riemann curvatre tensor of ds^2 2019 台大

3.