§ Immersion(浸射) Embedding(寢射)

$f: M \to N$ 是 local diffeomorphism at $p \in M$ 若

(1) dim M=dim N

(2) $(df)_p$ 是 diffeomorphically onto

則稱 $(df)_p : T_pM \to T_{f(p)}N$ 一 isomorphism(同構)

當 dim M<dim N

最好的情況是 $(df)_p$ 是 injective 此時 f 稱為在 p 的一個 immersion

(若對所有的 p 都成立 則 f 稱為一個 immersion)

定理

$f: M \to N$ 在 p 是一浸射 則在 p 有一 local coordinate 使得 f 是 canonical immersion 即 $(x^1, x^2, ..., x^m) \to (x^1, x^2, ..., x^m, 0, .., 0)$

當 $f: M \to N$ 在 p 是一浸射 且

(1) f 是同態(homeomorphism)onto f(M)

(2) with its subspace topology

則 f 稱為(differentiable)embedding

[DG12] p.13

Lemma 1.3.1 *Let $f: M \to N$ be an immersion, $\dim M = m, \dim N = n, x \in M$. Then there exist a neighborhood U of x and a chart (V, y) on N with $f(x) \in V$, such that*

(i) $f_{|U}$ is a differentiable embedding, and
(ii) $y^{m+1}(p) = \ldots = y^n(p) = 0$ for all $p \in f(U) \cap V$.

(proof followed)

If $f: M \to N$ is a differentiable embedding, $f(M)$ is called a *differentiable submanifold* of N. A subset N' of N, equipped with the relative topology, thus is a differentiable submanifold of N, if N' is a manifold and the inclusion is a differentiable embedding.

例

1. $f: R \to R^2$ $f(t) = (t^2, t^3)$ 在 t=0 不是 immersion
2. $f: R \to R^2$ f(t)=(cost,sin2t) 是 immersion 但不是 embedding
3. $f: R \to R^2$ $f(t) = (e^t \cos t, e^t \sin t)$ 是 embedding

若$M \subset N$且 inclusion map 是一 embedding 稱 M 是 N 的 submanifold

$f: M \to N$

若$(df)_p$是 surjective 則稱$p \in M$為 regular point(否則稱為 critical point)

若 $f^{-1}(q)$是 regular point 則稱$q \in N$為 regular value

Theorem 5.6 *Let $q \in N$ be a regular value of $f : M \to N$ and assume that the level set $L := f^{-1}(q) = \{p \in M \mid f(p) = q\}$ is nonempty. Then L is a submanifold of M and $T_pL = \ker(df)_p \subset T_pM$ for all $p \in L$.*

證明...

Theorem 5.7 (Whitney) *Any smooth manifold M of dimension n can be embedded in $\mathbb{R}^{2n}$ (and, provided that $n > 1$, immersed in $\mathbb{R}^{2n-1}$).* □

[DG001]p.26 習作

$S^n = \{x \in R^{n+1} | (x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2 = 1\}$

證明S^n是R^{n+1}的 n-dim submanifold 且$T_x S^n = \{v \in R^{n+1} | <x,v> = 0\}$

Consider the map $f : \mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$f(x^1, \ldots, x^n) = (x^1)^2 + \cdots + (x^{n+1})^2.$$

Its derivative

$$(df)_x = 2x^1 dx^1 + \cdots + 2x^{n+1} dx^{n+1}$$

is clearly injective for $x \neq 0$, as it is represented by the nonvanishing matrix

$$\left(2x^1 \mid \cdots \mid 2x^{n+1}\right).$$

Therefore, 1 is a regular value of f, and so $S^n = f^{-1}(1)$ is an n-dimensional manifold (cf. Theorem 5.6). Moreover, we have

$$\begin{aligned} T_x S^n = \ker(df)_x &= \{v \in T_x\mathbb{R}^{n+1} \mid (df)_x(v) = 0\} \\ &= \{v \in \mathbb{R}^{n+1} \mid x^1 v^1 + \cdots + x^{n+1} v^{n+1} = 0\} \\ &= \{v \in \mathbb{R}^{n+1} \mid \langle x, v \rangle = 0\}, \end{aligned}$$

where we have used the identification $T_x\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$.