

§ 三維球面

一直線與一圓的乘積 $I \times S$ 是圓柱。



1. 穩定的封閉 CMC 必為球面 J. L. Barbosa & M. do Carmo 1984

2. 宇宙可能 99% 是 S^3

(1)[Eleonora Di valentino](#) (2)[Alessandro Melchiorri](#) (3)[Joseph Silk](#) 宇宙空間曲率為正。

3. Poincare 猜想：任何一個單連通閉 3 維流形一定跟 S^3 拓撲等價(同胚)。

4. 在[陀螺](#)中又看到 S^3 : S^3 是 $SO(3)$ 的 universal covering manifold

5. The Ricci tensor is twice the metric $R_{\mu\nu} = 2g_{\mu\nu}$

$$S^3, x^\mu = (\psi, \theta, \phi), \quad ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$R_{ij} = (n-1)g_{ij} \quad \text{where } n=3, \quad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \psi & 0 \\ 0 & 0 & \sin^2 \psi \sin^2 \theta \end{pmatrix}$$

求

$$(a) \quad \Gamma^i_{jk} = \frac{1}{2} g^{il} \left\{ \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\}$$

(b) 求 Riemannian tensor, Ricci tensor, Ricci scalar

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$$

$$R^l_{ijk} = \Omega^l_i(E_j, E_k)$$

(c) Show that $R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)} (g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})$ is obeyed by this metric, confirming that the 3-sphere is a maximally symmetric space.

$$E_1 = \frac{\partial}{\partial \psi}, E_2 = \frac{1}{\sin \psi} \frac{\partial}{\partial \theta}, E_3 = \frac{1}{\sin \psi \sin \theta} \frac{\partial}{\partial \phi}$$

$$\omega^1 = d\psi, \quad \omega^2 = \sin \psi d\theta, \quad \omega^3 = \sin \psi \sin \theta d\phi$$

$$\text{Then } g = \sum_i \omega^i \otimes \omega^i$$

Cartan formula :

$$d\omega^i = \sum_j \omega^j \wedge \omega_j^i, \quad \omega_i^j + \omega_j^i = 0, \quad \Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$$

$$d\omega^1 = 0, d\omega^2 = \cos\psi d\psi \wedge d\theta, d\omega^3 = \cos\psi \sin\theta d\psi \wedge d\phi + \sin\psi \cos\theta d\theta \wedge d\phi$$

$$d\omega^1 = \omega^2 \wedge \omega_2^1 + \omega^3 \wedge \omega_3^1 = 0 \Rightarrow d\theta \wedge \omega_1^2 + \sin\theta d\phi \wedge \omega_1^3 = 0$$

$$d\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^2 = d\psi \wedge \omega_1^2 + \sin\psi \sin\theta d\phi \wedge \omega_3^2 = \cos\psi d\psi \wedge d\theta$$

$$\therefore \omega_1^2 = \cos\psi d\theta$$

$$d\omega^3 = \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = \cos\psi \sin\theta d\psi \wedge d\phi + \sin\psi \cos\theta d\theta \wedge d\phi$$

$$= d\psi \wedge \omega_1^3 + \sin\psi d\theta \wedge \omega_2^3$$

$$\therefore \omega_1^3 = \cos\psi \sin\theta d\phi, \quad \omega_2^3 = \cos\theta d\phi$$

$$\Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$$

$$\Omega_1^2 = d\omega_1^2 - \omega_1^3 \wedge \omega_3^2 = -\omega^1 \wedge \omega^2$$

$$\Omega_1^3 = d\omega_1^3 - \omega_1^2 \wedge \omega_2^3 = -\omega^1 \wedge \omega^3$$

$$\Omega_2^3 = d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = -\omega^2 \wedge \omega^3$$

$$R_{ijk}^l = \Omega_i^l(E_j, E_k), \quad R_{iji}^j = \Omega_i^j(E_j, E_i)$$

$$R_{212}^1 = \sin^2\psi, R_{221}^1 = -\sin^2\psi, \quad R_{313}^1 = \sin^2\psi \sin^2\theta, R_{331}^1 = -\sin^2\psi \sin^2\theta$$

$$R_{121}^2 = \Omega_1^2(E_1, E_2) = 1, \quad R_{112}^2 = -1, \quad R_{323}^2 = \sin^2\psi \sin^2\theta, R_{332}^2 = -\sin^2\psi \sin^2\theta$$

$$R_{131}^3 = \Omega_1^3(E_1, E_3) = 1, \quad R_{113}^3 = -1, \quad R_{232}^3 = \Omega_2^3(E_2, E_3) = \sin^2\psi, \quad R_{223}^3 = -\sin^2\psi$$

$$R_{\theta\psi\theta}^\psi = \frac{\partial \Gamma_{\theta\theta}^\psi}{\partial \psi} - \frac{\partial \Gamma_{\theta\psi}^\psi}{\partial \theta} + \Gamma_{\lambda\psi}^\psi \Gamma_{\theta\theta}^\lambda - \Gamma_{\lambda\theta}^\psi \Gamma_{\theta\psi}^\lambda, \quad \text{where } \lambda = \psi, \theta, \phi$$

= $\sin^2 \psi$ has been confirmed ◦

Then $R_{212}^1 = \Omega_2^1(E_1, E_2) = (\omega^1 \wedge \omega^2)(E_1, E_2) = ? \dots$

Cartan formula 的方法不知道哪裡出了錯!

正確的結果應該是

$$R_{ij} = (n-1)g_{ij} \quad \text{where } n=3, \quad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \psi & 0 \\ 0 & 0 & \sin^2 \psi \sin^2 \theta \end{pmatrix}$$

The variational principle provides a convenient way to actually calculate the Christoffel symbols for a given metric ◦

$$I = \frac{1}{2} \int f d\tau = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

$$ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$I = \frac{1}{2} \int \left[\left(\frac{d\psi}{d\tau} \right)^2 + \sin^2 \psi \left(\frac{d\theta}{d\tau} \right)^2 + \sin^2 \psi \sin^2 \theta \left(\frac{d\phi}{d\tau} \right)^2 \right] d\tau$$

$S(q) = \int L(t, q(t), \dot{q}(t)) dt$, If it independent of t , then the E-L equation are

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0, \quad \text{where } L = (\dot{\psi})^2 + \sin^2 \psi (\dot{\theta})^2 + \sin^2 \psi \sin^2 \theta (\dot{\phi})^2$$

For ψ , the E-L equation is $\frac{\partial L}{\partial \psi} - \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{\psi}} \right) = 0$

$$\frac{\partial L}{\partial \psi} = 2 \sin \psi \cos \psi (\dot{\theta})^2 + 2 \sin \psi \cos \psi \sin^2 \theta (\dot{\phi})^2, \quad \text{and} \quad \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{\psi}} \right) = \frac{d}{d\tau} (2\dot{\psi}) = 2\ddot{\psi}$$

$$2 \sin \psi \cos \psi (\dot{\theta})^2 + 2 \sin \psi \cos \psi \sin^2 \theta (\dot{\phi})^2 - 2\ddot{\psi} = 0,$$

$$\ddot{\psi} - \sin \psi \cos \psi (\dot{\theta})^2 - \sin \psi \cos \psi \sin^2 \theta (\dot{\phi})^2 = 0$$

The geodesic equation are $\ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0$, here $x^k = \psi$

The geodesic equation is $\ddot{\psi} + \Gamma_{\theta\theta}^{\psi} (\dot{\theta})^2 + \Gamma_{\phi\phi}^{\psi} (\dot{\psi})^2 = 0$

Thus we have $\Gamma_{\theta\theta}^{\psi} = -\sin\psi \cos\psi, \Gamma_{\phi\phi}^{\psi} = -\sin\psi \cos\psi \sin^2\theta$

For θ , the Euler equation is $\frac{\partial L}{\partial \theta} - \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0$

$$\frac{\partial L}{\partial \theta} = 2 \sin^2 \psi \sin \theta \cos \theta (\dot{\phi})^2$$

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{d}{d\tau} (\sin^2 \psi (2\dot{\theta})) = 2 \sin \psi \cos \psi \dot{\psi} (2\dot{\theta}) + 2 \sin^2 \psi \ddot{\theta}$$

$$\ddot{\theta} + 2 \cot \psi \dot{\theta} \dot{\psi} - \sin \theta \cos \theta (\dot{\phi})^2 = 0$$

Compare with $\ddot{\theta} + \Gamma_{ij}^{\theta} \dot{x}^i \dot{x}^j = 0$,

We have $\Gamma_{\psi\theta}^{\theta} = \Gamma_{\theta\psi}^{\theta} = \cot \psi, \Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta$

Again for ϕ , the Euler equation is $\frac{\partial L}{\partial \phi} - \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = 0$, we have

$$\ddot{\phi} + 2 \cot \psi \dot{\psi} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0$$

So $\Gamma_{\psi\phi}^{\phi} = \Gamma_{\phi\psi}^{\phi} = \cot \psi, \Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$

We have $\Gamma_{\theta\theta}^{\psi} = -\sin \psi \cos \psi, \Gamma_{\phi\phi}^{\psi} = -\sin \psi \cos \psi \sin^2 \theta$

$$\Gamma_{\psi\theta}^{\theta} = \Gamma_{\theta\psi}^{\theta} = \cot \psi, \Gamma_{\phi\phi}^{\theta} = \Gamma_{\phi\theta}^{\theta} = -\sin \theta \cos \theta$$

$$\Gamma_{\psi\phi}^{\phi} = \Gamma_{\phi\psi}^{\phi} = \cot \psi, \Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$$

(b) The Riemann tensor components are

$$R_{\sigma\mu\nu}^{\rho} = \partial_{\mu} \Gamma_{\nu\sigma}^{\rho} - \partial_{\nu} \Gamma_{\mu\sigma}^{\rho} + \Gamma_{\mu\lambda}^{\rho} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\rho} \Gamma_{\mu\sigma}^{\lambda}$$

$$R_{\theta\psi\theta}^{\psi} = \sin^2 \psi, R_{\theta\psi\psi}^{\psi} = -\sin^2 \psi, R_{\phi\psi\phi}^{\psi} = \sin^2 \psi \sin^2 \theta, R_{\phi\psi\psi}^{\psi} = -\sin^2 \psi \sin^2 \theta$$

$$R_{\psi\nu\theta}^{\theta} = -1, R_{\psi\theta\nu}^{\theta} = 1, R_{\phi\theta\phi}^{\theta} = \sin^2 \psi \sin^2 \theta, R_{\phi\phi\theta}^{\theta} = -\sin^2 \psi \sin^2 \theta$$

$$R_{\psi\nu\phi}^{\phi} = -1, R_{\psi\phi\nu}^{\phi} = 1, R_{\theta\theta\phi}^{\phi} = -\sin^2 \psi, R_{\theta\phi\theta}^{\phi} = \sin^2 \psi$$

例如

$$\begin{aligned} R_{\phi\nu\phi}^{\psi} &= \partial_{\nu} \Gamma_{\phi\phi}^{\psi} - \partial_{\phi} \Gamma_{\nu\phi}^{\psi} + \Gamma_{\nu\lambda}^{\psi} \Gamma_{\phi\phi}^{\lambda} - \Gamma_{\phi\lambda}^{\psi} \Gamma_{\nu\phi}^{\lambda} \\ &= \partial_{\psi} (-\sin \psi \cos \psi \sin^2 \theta) - \partial_{\phi} 0 + 0 - (-\sin \psi \cos \psi \sin^2 \theta)(\cot \psi) \\ &= -\cos^2 \psi \sin^2 \theta + \sin^2 \psi \sin^2 \theta \cot^2 \psi = \sin^2 \psi \end{aligned}$$

$$\begin{aligned} R_{\psi\theta\nu}^{\theta} &= \partial_{\theta} \Gamma_{\psi\nu}^{\theta} - \partial_{\nu} \Gamma_{\psi\theta}^{\theta} + \Gamma_{\theta\lambda}^{\theta} \Gamma_{\psi\nu}^{\lambda} - \Gamma_{\nu\lambda}^{\theta} \Gamma_{\psi\theta}^{\lambda} \\ &= -\partial_{\psi} (\csc^2 \psi) - (\psi \cot \psi) \csc^2 \psi = \csc^2 \psi - \cot^2 \psi = 1 \end{aligned}$$

$$\begin{aligned} R_{\theta\theta\phi}^{\phi} &= \partial_{\theta} \Gamma_{\theta\phi}^{\phi} - \partial_{\phi} \Gamma_{\theta\theta}^{\phi} + \Gamma_{\theta\lambda}^{\phi} \Gamma_{\theta\phi}^{\lambda} - \Gamma_{\phi\lambda}^{\phi} \Gamma_{\theta\theta}^{\lambda} \\ &= -\csc^2 \theta + \cot^2 \theta - \cot^2 \theta = -\csc^2 \theta \end{aligned}$$

$$R_{\nu\nu} = R_{\psi\lambda\nu}^{\lambda} = 1+1=2, R_{\theta\theta} = R_{\theta\lambda\theta}^{\lambda} = \sin^2 \psi + \sin^2 \psi = 2\sin^2 \psi$$

$$R_{\phi\phi} = R_{\phi\lambda\phi}^{\lambda} = \sin^2 \psi \sin^2 \theta + \sin^2 \psi \sin^2 \theta = 2\sin^2 \psi \sin^2 \theta$$

The Ricci tensor is twice the metric $R_{\mu\nu} = 2g_{\mu\nu}$

The Ricci scalar $R = g^{\mu\mu} R_{\mu\mu} = 6$

(c) Show that $R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$ is obeyed by this metric, confirming that the 3-sphere is a maximally symmetric space.

[Foundation of Differential Geometry VII] by Kobayashi Nomizu

A Riemannian manifold is called Einstein if $S = \rho g$, where S is the Ricci tensor and ρ is a constant.

p.35 Let M be a hypersurface immersed in R^{n+1} , at each point of M , the Ricci tensor S is given by $S(X, Y) = g(AX, Y) \text{trace} A - g(A^2 X, Y)$, $X, Y \in T_x(M)$

For $n \geq 3$, if M is Einstein then $\rho \geq 0$

(1) $\rho = 0 \dots$

(2) $\rho > 0$ M is locally a hypersphere

參考

1. [Spacetime and Geometry] Ch3 EX08 and EX16

2. [RG4102]--- $I \times S^2$ $g = A^2(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$

S^2 做為一個 Riemannian manifold $g = d\theta^2 + \sin^2 \theta d\phi^2$ induced from R^3

3. [Wormhole metric](#) by Ellis

$c^2 dt^2 = d\rho^2 + (\rho^2 + n^2)(d\theta^2 + \sin^2 \theta d\phi^2)$ where n is the drainhole parameter

4. $ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2 \theta d\phi^2)$ by MT wormhole

5. [Everything Wormhole](#)

6. [Ricci curvature](#)

7. Geometry of 3-Sphere [Garret Sobczyk](#) [What's a Pauli matrix]
[Geometric Algebra---Spinors]

8. [三維球面](#) 平斯