

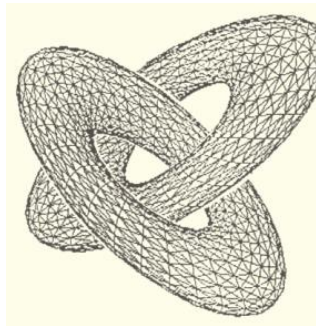
§ 三維球面

§ 01 前言

§ 02 Euler equation Geodesic equation Christoffel symbols Rieman tensor Ricci tensor

The Ricci tensor $R_{\mu\nu} = 2g_{\mu\nu}$, $ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$ Cartan formula : $d\omega^i = \sum_j \omega^j \wedge \omega_j^i$, $\omega_i^j + \omega_j^i = 0$, $\Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$ Einstein manifold if $Ric = \rho g$, where ρ is a constant . S^3 隨 Ricci flow 演進。§ 03 $I \times S^2$ § 04 與 $SU(2)$, $SO(3)$ 的關係

§ 01 前言



先瞄一下平斯先生如何介紹[三維球面](#)，文中說道把 S^3 解構成兩個環體，再把它們黏再一起…竟然與 $SO(3)$ 還有關係，倉西正武(Masatake Karunishi)又是誰？

Poincare 猜想：任何一個單連通閉 3 維流形一定跟 S^3 拓撲等價(同胚)。

宇宙可能 99% 是 S^3 ，義大利天體物理學家范倫提諾小姐([Eleonora Di valentino](#))如是說。

[[ResearchGate](#)] [Challenges to the [Standard Cosmological Model](#)] (2)[Alessandro Melchiorri](#) (3)[Joseph Silk](#) 宇宙空間曲率為正。

DeepSeek 回答：

In cosmology , the 3-sphere is used as a model for the shape of the universe in certain solutions to Einstein equations . It also appears in the study of gauge theories and string theory .

DeepSeek 回答：

The 3-sphere can be identified with the group of unit quaternions , which makes it a Lie group . This is important in the study of rotations in three dimensions (via the double cover $S^3 \rightarrow SO(3)$)

最後 DeepSeek 說：The 3-sphere is a fascinating object that bridges geometry , topology , and physics and it continues to be an active area of research .

§ 02 The Ricci tensor is twice the metric $R_{\mu\nu} = 2g_{\mu\nu}$
 $S^3, x^\mu = (\psi, \theta, \phi)$, $ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$

$$R_{ij} = (n-1)g_{ij} \quad \text{where } n=3, \quad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \psi & 0 \\ 0 & 0 & \sin^2 \psi \sin^2 \theta \end{pmatrix}$$

求

$$(a) \quad \Gamma_{jk}^i = \frac{1}{2} g^{il} \left\{ \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} \quad \text{by Euler equation and geodesics}$$

$$S(q) = \int L(t, q(t), \dot{q}(t)) dt$$

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0$$

$$ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$L = (\dot{\psi})^2 + \sin^2 \psi (\dot{\theta})^2 + \sin^2 \psi \sin^2 \theta (\dot{\phi})^2$$

$$\text{For } \theta, \text{ the Euler equation is } \frac{\partial L}{\partial \theta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0$$

$$\frac{\partial L}{\partial \theta} = 2 \sin^2 \psi \sin \theta \cos \theta (\dot{\phi})^2$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{d}{dt} (\sin^2 \psi (2\dot{\theta})) = 2 \sin \psi \cos \psi \dot{\psi} (2\dot{\theta}) + 2 \sin^2 \psi \ddot{\theta}$$

$$\ddot{\theta} + 2 \cot \psi \dot{\theta} \dot{\psi} - \sin \theta \cos \theta (\dot{\phi})^2 = 0$$

$$\text{Compare with } \ddot{\theta} + \Gamma_{ij}^{\theta} \dot{x}^i \dot{x}^j = 0,$$

$$\text{We have } \Gamma_{\psi\theta}^{\theta} = \Gamma_{\theta\psi}^{\theta} = \cot \psi, \Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta$$

$$\text{Again for } \phi, \text{ the Euler equation is } \frac{\partial L}{\partial \phi} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = 0, \text{ we have}$$

$$\ddot{\phi} + 2 \cot \psi \dot{\psi} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0, \quad \Gamma_{\psi\phi}^{\phi} = \Gamma_{\phi\psi}^{\phi} = \cot \psi, \Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$$

We have

$$\Gamma_{\theta\theta}^{\psi} = -\sin \psi \cos \psi, \Gamma_{\phi\phi}^{\psi} = -\sin \psi \cos \psi \sin^2 \theta$$

$$\Gamma_{\psi\theta}^\theta = \Gamma_{\theta\psi}^\theta = \cot \psi, \Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta$$

$$\Gamma_{\psi\phi}^\phi = \Gamma_{\phi\psi}^\phi = \cot \psi, \Gamma_{\theta\phi}^\phi = \Gamma_{\phi\theta}^\phi = \cot \theta$$

(b) 求 Riemannian tensor, Ricci tensor, Ricci scalar

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda$$

$$R_{\psi\psi\theta}^\psi = \sin^2 \psi, R_{\theta\theta\psi}^\psi = -\sin^2 \psi, R_{\phi\psi\phi}^\psi = \sin^2 \psi \sin^2 \theta, R_{\phi\phi\psi}^\psi = -\sin^2 \psi \sin^2 \theta$$

$$R_{\psi\psi\theta}^\theta = -1, R_{\psi\theta\psi}^\theta = 1, R_{\phi\theta\phi}^\theta = \sin^2 \psi \sin^2 \theta, R_{\phi\phi\theta}^\theta = -\sin^2 \psi \sin^2 \theta$$

$$R_{\psi\psi\phi}^\phi = -1, R_{\psi\phi\psi}^\phi = 1, R_{\theta\theta\phi}^\phi = -\sin^2 \psi, R_{\theta\phi\theta}^\phi = \sin^2 \psi$$

例如

$$R_{\phi\psi\phi}^\psi = \partial_\psi \Gamma_{\phi\phi}^\psi - \partial_\phi \Gamma_{\psi\phi}^\psi + \Gamma_{\psi\lambda}^\psi \Gamma_{\phi\phi}^\lambda - \Gamma_{\phi\lambda}^\psi \Gamma_{\psi\phi}^\lambda$$

$$= \partial_\psi (-\sin \psi \cos \psi \sin^2 \theta) - \partial_\phi 0 + 0 - (-\sin \psi \cos \psi \sin^2 \theta)(\cot \psi)$$

$$= -\cos^2 \psi \sin^2 \theta + \sin^2 \psi \sin^2 \theta \cot \psi = \sin^2 \psi \sin^2 \theta$$

$$R_{\psi\theta\psi}^\theta = \partial_\theta \Gamma_{\psi\psi}^\theta - \partial_\psi \Gamma_{\psi\theta}^\theta + \Gamma_{\theta\lambda}^\theta \Gamma_{\psi\psi}^\lambda - \Gamma_{\psi\lambda}^\theta \Gamma_{\psi\theta}^\lambda$$

$$= -\partial_\theta (\cot \psi) - (\psi \cot \psi) = \csc^2 \psi - \cot^2 \psi = 1$$

$$R_{\theta\theta\phi}^\phi = \partial_\theta \Gamma_{\theta\phi}^\phi - \partial_\phi \Gamma_{\theta\theta}^\phi + \Gamma_{\theta\lambda}^\phi \Gamma_{\theta\phi}^\lambda - \Gamma_{\phi\lambda}^\theta \Gamma_{\theta\theta}^\lambda$$

$$= -\csc^2 \theta + \cot^2 \theta - \cot^2 \theta - (\sin \psi \cos \psi)^2 \psi = -\sin^2 \psi$$

$$R_{\psi\psi} = R_{\psi\lambda\psi}^\lambda = 1 + 1 = 2, R_{\theta\theta} = R_{\theta\lambda\theta}^\lambda = \sin^2 \psi + \sin^2 \psi = 2 \sin^2 \psi$$

$$R_{\phi\phi} = R_{\phi\lambda\phi}^\lambda = \sin^2 \psi \sin^2 \theta + \sin^2 \psi \sin^2 \theta = 2 \sin^2 \psi \sin^2 \theta$$

The Ricci tensor is twice the metric $R_{\mu\nu} = 2g_{\mu\nu}$

The Ricci scalar $R = g^{\mu\mu} R_{\mu\mu} = 6$

ChatGPT 給的答案：

S^3 可以看作是 R^4 的超曲面，它具有標準的 $SU(2)$ 左不變量，也就是 Hopf 瓶射誘導的度量，可以寫成 $ds^2 = \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ ，其中 σ_i 是 $SU(2)$ 的左不變 1-form，

滿足 Maurer-Cartan 結構方程：
$$d\sigma_i = -\frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$$

選擇 S^3 上的正交標架 $\{e_1, e_2, e_3\}$ ，使得其對偶 1-form 為 $\{\sigma_1, \sigma_2, \sigma_3\}$ ，即

$$g(e_i, e_j) = \delta_{ij}$$

計算 Levi-Civita connection 1-form ω_{ij} ，由結構方程 $d\sigma_i + \sum_j \omega_{ij} \wedge \sigma_j = 0$ 可以解得

$$\omega_{12} = -\frac{1}{2}\sigma_3, \omega_{23} = -\frac{1}{2}\sigma_1, \omega_{31} = -\frac{1}{2}\sigma_2$$

計算曲率 2-form Ω_{ij} ：
$$\Omega_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj}$$

$$d\omega_{12} = -\frac{1}{2}d\sigma_3 = \frac{1}{4}\sigma_1 \wedge \sigma_2$$

$$\omega_{12} \wedge \omega_{23} = \left(-\frac{1}{2}\sigma_3\right) \wedge \left(-\frac{1}{2}\sigma_1\right) = \frac{1}{4}\sigma_3 \wedge \sigma_1$$

$$\text{因此 } \Omega_{12} = \frac{1}{4}(\sigma_1 \wedge \sigma_2 + \sigma_3 \wedge \sigma_1) = \frac{1}{4}\sigma_1 \wedge \sigma_2$$

$$\text{同理 } \Omega_{23} = \frac{1}{4}\sigma_2 \wedge \sigma_3, \Omega_{31} = \frac{1}{4}\sigma_3 \wedge \sigma_1$$

計算 Ricci 曲率張量：
$$Ric(e_i, e_j) = \sum_k R_{ikjk}$$

$$\text{由於 } \Omega_{ij} = \frac{1}{4}\sigma_i \wedge \sigma_j, \text{ 可以得到 } R_{ikjk} = \frac{1}{4}$$

$$\text{所以 } Ric(e_i, e_j) = \sum_k g^{kk} R_{ikjk} = \frac{1}{4} \sum_k g_{kk} \delta_{ij} = \frac{3}{4} \delta_{ij}$$

計算 Ricci scalar curvature：

$$R = \sum_{i,j} g^{ij} Ric(e_i, e_j), \text{ 由於 } g^{ij} = \delta^{ij}, R = \sum_i \frac{3}{4} = 3 \times \frac{3}{4} = \frac{9}{4}$$

結論

對於單位半徑的 S^3 ，Ricci 曲率張量為 $Ric = \frac{3}{4}g$

如果我們使用一般半徑 r 的 3-sphere，則可以通過適當的縮放得到 $Ric = \frac{2}{r^2}g$

特別地，對於標準單位球面 S^3 ，這符合常見的結果 $Ric = 2g$

§ 03 $I \times S^2$



一直線與一圓的乘積 $I \times S$ 是圓柱。

§ From $I \times S^2$ to S^3

The unit $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ with the Riemannian metric induced from \mathbb{R}^4 . Write $x^2 + y^2 + z^2 = 1 - w^2$, and use the spherical coordinates for the slices of radius $r = \sqrt{1 - w^2}$

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \\ w = \sqrt{1 - r^2} \end{cases}$$

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad \text{and} \quad dw = \frac{-r}{\sqrt{1 - r^2}} dr, \quad dw^2 = \frac{r^2}{1 - r^2} dr^2$$

$$\text{因此 } dx^2 + dy^2 + dz^2 + dw^2 = \frac{1}{1 - r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

The induced metric of S^3 from \mathbb{R}^4 is $g = \frac{1}{1 - r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$

From $I \times S^2$, $A(r) = (1 - r^2)^{-\frac{1}{2}}$, $M = I \times S^2$ is locally isometric to S^3

[N3802]

$$R_{11} = \Omega_1^2(E_2, E_1) + \Omega_1^3(E_3, E_1) = \frac{A'}{rA^3} + \frac{A'}{rA^3} = \frac{2A'}{rA^3} = \dots = 2 \quad \text{直接計算}$$

$$R_{22} = \Omega_2^1(E_1, E_2) + \Omega_2^3(E_3, E_2) = \frac{A'}{rA^3} + \frac{1}{r^2} \left(1 - \frac{1}{A^2}\right) = \dots = 1 + 1 = 2 \quad \text{直接計算}$$

$$R_{33} = \Omega_3^1(E_1, E_3) + \Omega_3^2(E_2, E_3) = \frac{A'}{rA^3} + \frac{1}{r^2} \left(1 - \frac{1}{A^2}\right) = \dots = 2$$

$R_{ij} = 2\delta_{ij}$ in an orthonormal frame.

Ask : when we compute the Ricci tensor of 3-sphere from the induced metric and the metric of R^4 , the forms of the Ricci tensor are different ?

DeepSeek answers :

1. Ricci Tensor of S^3 from the Induced Metric

The 3-sphere is S^3 a Riemannian manifold with a constant positive curvature.

Its induced metric (from R^4) is the round metric, and the Ricci tensor for S^3 is proportional to the metric itself.

Specifically, for a 3-sphere of radius a , the Ricci tensor is : $R_{ij} = \frac{2}{a^2} g_{ij}$

Where g_{ij} is the metric on S^3 . This reflects the fact that S^3 is an Einstein manifold.

2. Ricci Tensor from the Metric of R^4

The 3-sphere S^3 is embedded in R^4 , and its metric is induced from the Euclidean metric of R^4 .

In R^4 , the Ricci tensor vanishes identically because R^4 is flat : $R_{\mu\nu} = 0$ (in R^4).

However, when you restrict to the 3-sphere S^3 , the extrinsic curvature (second fundamental form) and the Gauss-Codazzi equations come into play. The Ricci tensor of S^3 arises from the intrinsic curvature of the submanifold, not from the curvature of R^4 .

3. Why the Forms Might Appear Different

- (1) Intrinsic vs. Extrinsic Viewpoint: When computing the Ricci tensor intrinsically (using the metric of S^3), you directly work with the geometry of S^3 . When working extrinsically (using the embedding in R^4), you might use the Gauss-Codazzi equations, which relate the curvature of the submanifold to the curvature of the ambient space and the extrinsic curvature.
- (2) Coordinate Systems: The form of the Ricci tensor might look different depending on the coordinates used. For example, in spherical coordinates on S^3 , the metric and Ricci tensor have specific forms, while in Cartesian coordinates in R^4 , the embedding of S^3 is described differently.

4. Consistency

Despite the apparent differences in form, the Ricci tensor of S^3 computed intrinsically (from the induced metric) and extrinsically (from the embedding in R^4) must agree.

The key point is that the intrinsic curvature of S^3 is determined by its embedding in R^4 , and both methods should yield the same result.

5. Summary

The Ricci tensor of S^3 is $R_{ij} = \frac{2}{a^2} g_{ij}$, whether computed intrinsically or extrinsically.

The forms might appear different due to the choice of coordinates or the method of computation, but they are fundamentally the same.

§ [RG01] p.81 有一個有意思的習作

$$S^3 = \{(x, y, z, w) \in R^4 \mid x^2 + y^2 + z^2 + w^2 = 2\}$$

$$T^2 = \{(x, y, z, w) \in R^4 \mid x^2 + y^2 = z^2 + w^2 = 1\}$$

The submanifold $T^2 \subset S^3$ splits S^3 into two connected components.

Let M be one of these components and let $\omega = zdx \wedge dy \wedge dw - xdy \wedge dz \wedge dw$ to be a 3-form.

Compute the two possible value of $\int_M \omega$

Stokes theorem

Let M be an n -dimensional oriented smooth manifold with boundary, let ω be a $(n-1)$ -differential form on M with compact support, and let $i: \partial M \rightarrow M$ be the inclusion of the boundary ∂M in M .

Then $\int_{\partial M} i^* \omega = \int_M d\omega$, where we consider ∂M with the induced orientation.

induced orientation.

以下有 Stokes 定理的證明。

T^2 divides S^3 into **two connected components**

$$M_1 := \{(x, y, z, w) \in S^3 \mid x^2 + y^2 < 1\} \text{ and } M_2 := \{(x, y, z, w) \in S^3 \mid x^2 + y^2 > 1\}$$

Let $\varphi_i: U_i \rightarrow M_i (i=1, 2)$ defined by

$$\varphi_i(r, v, w) = (r \cos v, r \sin v, \sqrt{2-r^2} \cos v, \sqrt{2-r^2} \sin v) \text{ on}$$

$$U_1 := (0,1) \times (0, 2\pi) \times (0, 2\pi) \text{ and } U_2 := (1, \sqrt{2}) \times (0, 2\pi) \times (0, 2\pi) \text{ . Then}$$

$$\varphi_i^* dx = \cos v dr - r \sin v dv$$

$$\varphi_i^* dy = \sin v dr + r \cos v dv$$

$$\varphi_i^* dz = -\frac{r}{\sqrt{2-r^2}} \cos v dr - \sqrt{2-r^2} \sin v dv$$

$$\varphi_i^* dw = -\frac{r}{\sqrt{2-r^2}} \sin v dr + \sqrt{2-r^2} \cos v dv$$

$$\begin{aligned} \text{And so } \varphi_i^* \omega &= (\sqrt{2-r^2} \cos v)(\varphi_i^* dx \wedge \varphi_i^* dy \wedge \varphi_i^* dw) - (r \cos v)(\varphi_i^* dy \wedge \varphi_i^* dz \wedge \varphi_i^* dw) \\ &= (r(2-r^2) \cos^2 v - r^3 \cos^2 v) dr \wedge dv \wedge dw \\ &\dots \end{aligned}$$

Since the sets $M_1 / \varphi_1(U_1), M_2 / \varphi_2(U_2)$ have measure zero , we have

$$\begin{aligned} \int_{M_1} \omega &= \pm \int_{U_1} \varphi_1^* \omega = \pm \int_0^1 \int_0^{2\pi} \int_0^{2\pi} (r(2-r^2) \cos^2 v - r^3 \cos^2 v) dv dv dr = \pm \pi^2 \\ \int_{M_2} \omega &= \pm \int_{U_2} \varphi_2^* \omega = \pm \int_1^{\sqrt{2}} \int_0^{2\pi} \int_0^{2\pi} (r(2-r^2) \cos^2 v - r^3 \cos^2 v) dv dv dr = \mp \pi^2, \end{aligned}$$

where the \pm signs depend on the choice of orientation.

在[RG01] p.84 Exercise 5.3 (1) Stokes theorem 的應用。

$$\S \text{ 特殊酉群 } SU(2) = \left\{ \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix} \middle| a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

我們可以透過四元數來看到 $SU(2)$ 與 S^3 視同胚的。

四元數 $q = x_0 + x_1 i + x_2 j + x_3 k$, $|q|=1$ 形成一個群 , 這恰好就是 S^3 ,

而四元數的矩陣表示恰好對應於 $SU(2)$, 因此 , S^3 具有李群結構 , 並且與 $SU(2)$ 拓撲同構。

在李群 $SU(2)$ 上 , 我們可以考慮左不變的(left-invariant)結構。特別是左不變向量場(Lie 代數生成元)。

$SU(2)$ 的李代數 $\mathfrak{su}(2)$ 由所有跡為零的反厄米 2×2 矩陣組成 , 其標準基底為

$$T_1 = \frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, T_2 = \frac{i}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, T_3 = \frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

滿足 $\mathfrak{su}(2)$ 的交換關係： $[T_1, T_2] = T_3, [T_2, T_3] = T_1, [T_3, T_1] = T_2$

這與李代數 $\mathfrak{so}(3)$ 的交換關係相同，表明 $SU(2)$ 是 $SO(3)$ 的二重覆蓋。

在 $SU(2)$ 上，我們可以透過左乘作用推廣這些基底來定義左不變向量場。

設 $g \in SU(2)$ ，則左乘 $L_g : h \rightarrow gh$ 推動李代數 $\mathfrak{su}(2)$ 的基底，形成左不變向量場。

若改用四元數表示，則左不變向量場可寫為：

$$X_1 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} + x_0 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_0}$$

$$X_2 = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} + x_0 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_0}$$

$$X_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_0 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_0}$$

這些向量場對應於 $\mathfrak{su}(2)$ 的基底，並在 S^3 上生成左不變結構。

應用與幾何意義

1. 黎曼幾何

S^3 配上標準的度量，成為均勻空間，並且左不變向量場對應於 Killing 向量場，表示球面的對稱性。

2. 物理應用

$SU(2)$ 出現在量子力學與粒子物理中，特別是自旋 $\frac{1}{2}$ 系統與弱交互作用。

3. 聯絡與纖維叢

在愛因斯坦時空理論中， S^3 經常作為 $SO(4)$ 的子群出現，並與 Hopf 纖維結構相關。這些結構使得 S^3 與 $SU(2)$ 在數學與物理中有豐富的應用。

在[陀螺](#)中又看到 S^3 ： S^3 是 $SO(3)$ 的 **universal covering manifold**，怎麼又與李群扯上關係了？

陀螺的旋轉與 $SO(3)$

陀螺的運動通常由**進動**（precession）、**章動**（nutation）和**自旋**（spin）組成：

1. **自旋**：陀螺繞自身軸的旋轉。
2. **進動**：陀螺軸繞固定軸（通常是重力方向）的緩慢旋轉。
3. **章動**：陀螺軸相對於進動軸的細微擺動。

這些運動可以用 $SO(3)$ 群中的元素來描述。 $SO(3)$ 的每個元素對應於三維空間中的一個旋轉，而陀螺的姿態可由一個時間相關的 $SO(3)$ 矩陣 $R(t)R(t)R(t)$ 來表示。

$$SO(3) \cong S^3 / Z_2$$

這表示四維球面（單位四元數的集合）在對應於正負四元數的等價類下，恰好形成 $SO(3)$ 群。這種對應源於四元數與三維旋轉的關係：

- 單位四元數 $q = a + bi + cj + dk$ 可以表示三維旋轉，這種表示方式比 $SO(3)$ 的矩陣形式更具數值穩定性（避免萬向鎖）。
- 四元數的乘法對應於旋轉的組合，因此 S^3 上的結構與旋轉群的組成方式一致。

由於四元數表示具有雙射性（每個旋轉對應兩個單位四元數 q 和 $-q$ ），所以 $SO(3)$ 與 S^3 之間的關係是兩者之間的雙覆蓋（double cover）。這種性質與自旋群 $SU(2)$ 亦相關，因為：

$$SU(2) \cong S^3$$

而 $SU(2)$ 也是 $SO(3)$ 的雙覆蓋。

3. 陀螺運動的數學描述

陀螺的運動由剛體動力學方程描述，通常用歐拉角（Euler angles）或旋轉矩陣 $R(t)$ 來表達：

1. Euler 方程（剛體旋轉方程）：

$$I\dot{\omega} + \omega \times (I\omega) = \tau$$

其中 ω 是角速度， I 是慣性張量， τ 是外力矩（如重力導致的力矩）。

2. 旋轉矩陣演化方程：

$$\dot{R}(t) = R(t)\Omega(t)$$

其中 $\Omega(t)$ 是角速度的斜對稱矩陣：

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

這個矩陣來自 $SO(3)$ 李代數 $\mathfrak{so}(3)$ 。

3. 四元數表示的運動方程（更穩定）：

$$\dot{q} = \frac{1}{2}q \cdot \omega^*$$

其中 q 是單位四元數， ω^* 是表示角速度的純虛四元數。這種表示方式避免了歐拉角中的特異點問題。

4. 總結

- 陀螺的旋轉運動可以用 $SO(3)$ 群來描述，這個群控制了三维剛體的旋轉。
- $SO(3)$ 與三球面 S^3 密切相關，因為 S^3 透過四元數提供了 $SO(3)$ 的雙覆蓋表示。
- 使用四元數可以避免旋轉矩陣或歐拉角的數值問題，使得模擬陀螺運動更穩定。
- 陀螺的動力學方程可以透過 $SO(3)$ 李代數 $\mathfrak{so}(3)$ 來理解，並可轉換為四元數或李群方法來求解。

這些數學結構不僅對於分析陀螺有用，也在機器人學、計算機圖形學、物理模擬等領域廣泛應用。

§ 06 參考資料

1. [Spacetime and Geometry] Ch3 EX08 and EX16

2. $I \times S^2$, $g = A^2(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$

S^2 做為一個 Riemannian manifold $g = d\theta^2 + \sin^2 \theta d\phi^2$ induced from R^3

3. [Wormhole metric](#) by Ellis

$c^2 dt^2 = d\rho^2 + (\rho^2 + n^2)(d\theta^2 + \sin^2 \theta d\phi^2)$ where n is the drainhole parameter

4. $ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2 \theta d\phi^2)$ by MT wormhole

5. [Everything Wormhole](#)

6. Geometry of 3-Sphere [Garret Sobczyk](#) [What is a Pauli matrix]
[Geometric Gagebra---Spinors]