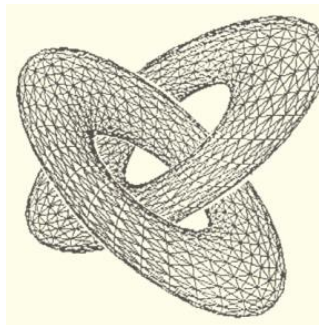


§ 三維球面

§ 01 前言



先瞄一下平斯先生如何介紹[三維球面](#)，文中說到把 S^3 解構成兩個環體，再把它們黏在一起…竟然與 $SO(3)$ 有關係。

1. 倉西正武([Masatake Karunishi](#) 1924-2021)
2. [[Thomas Banchoff](#)] [[ResearchGate](#)]
3. 宇宙可能 99% 是 S^3 ，義大利天體物理學家范倫提諾小姐([Eleonora Di valentino](#))如是說。(research of dark

energy)...

[[ResearchGate](#)] [Challenges to the [Standard Cosmological Model](#)] (2)[Alessandro Melchiorri](#) (3)[Joseph Silk](#) 宇宙空間曲率為正。

如果整個宇宙在[緩慢旋轉](#)，就能解決 Hubble tension problem。

§ 02

§ 03 Ricci tensor

§ 04 Geodesics

§ 05 Minimal surface on S^3 Minimal surface on S^3 : A survey of recent results Simon Brendle

§ 06 Vector fields on S^3

§ 07 S^3 上的 Lie groups $SO(3)$ $SU(2)$ $I \times S^2$

§ 08 Eigenvalue of Laplace-Beltrami operator

§ 09 Spectrum

§ 010 $H_{ol}(S^3) = SO(3)$

$$1. \quad ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$2. \quad \text{The Ricci tensor } R_{\mu\nu} = 2g_{\mu\nu}$$

Cartan structure equations :

$$(1) \quad d\omega^i = \sum_j \omega^j \wedge \omega_j^i \quad (2) \quad \Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$$

3. Poincare 猜想：任何一個單連通閉 3 維流形一定跟 S^3 拓撲等價(同胚)。
4. 與 $SU(2)$ ， $SO(3)$ 的關係

S^3 與 $SO(3)$ 之間有密切的關係，主要體現在拓撲結構與群結構上：

1. S^3 做為 $SU(2)$ 的流形結構
2. $SO(3)$ 與 $SU(2)$ 上的商群結構

$\varphi: SU(2) \rightarrow SO(3)$ 是群同態，其核為中心子群 $\{\pm I\}$ 。 $SO(3) \cong SU(2)/\{\pm I\}$ 也就是說， $SU(2)$ 中的兩個矩陣 U 和 $-U$ 在映射到 $SO(3)$ 時會得到相同的旋轉矩陣。

3. 拓撲與同倫性質

由於 S^3 是單連通的，而 $SO(3)$ 不是（ $SO(3)$ 的基本群為 $\mathbb{Z}/2\mathbb{Z}$ ），這進一步證明了 S^3 是 $SO(3)$ 的雙覆空間（double cover）。

這也解釋了為何在三維旋轉中會出現「旋量」（spinor）這種性質：旋量變換實際上是 $SU(2)$ 而非 $SO(3)$ 的表示。

§ 02 The Ricci tensor is twice the metric $R_{\mu\nu} = 2g_{\mu\nu}$

$$S^3, x^\mu = (\psi, \theta, \phi), \quad ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$R_{ij} = (n-1)g_{ij} \quad \text{where } n=3, \quad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \psi & 0 \\ 0 & 0 & \sin^2 \psi \sin^2 \theta \end{pmatrix}$$

求

$$(a) \quad \Gamma_{jk}^i = \frac{1}{2} g^{il} \left\{ \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} \quad \text{by Euler equation and geodesics}$$

$$S(q) = \int L(t, q(t), \dot{q}(t)) dt$$

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = 0$$

$$ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$L = (\dot{\psi})^2 + \sin^2 \psi (\dot{\theta})^2 + \sin^2 \psi \sin^2 \theta (\dot{\phi})^2$$

$$\text{For } \theta, \text{ the Euler equation is } \frac{\partial L}{\partial \theta} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = 0$$

$$\frac{\partial L}{\partial \theta} = 2 \sin^2 \psi \sin \theta \cos \theta (\dot{\phi})^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(\sin^2 \psi (2\dot{\theta})) = 2\sin \psi \cos \psi \dot{\psi} (2\dot{\theta}) + 2\sin^2 \psi \ddot{\theta}$$

$$\ddot{\theta} + 2\cot \psi \dot{\theta} \dot{\psi} - \sin \theta \cos \theta (\dot{\phi})^2 = 0$$

$$\text{Compare with } \ddot{\theta} + \Gamma_{ij}^{\theta} \dot{x}^i \dot{x}^j = 0 \quad ,$$

$$\text{We have } \Gamma_{\psi\theta}^{\theta} = \Gamma_{\theta\psi}^{\theta} = \cot \psi, \Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta$$

$$\text{Again for } \phi \quad , \quad \text{the Euler equation is } \frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \quad , \quad \text{we have}$$

$$\ddot{\phi} + 2\cot \psi \dot{\psi} \dot{\phi} + 2\cot \theta \dot{\theta} \dot{\phi} = 0 \quad , \quad \Gamma_{\psi\phi}^{\phi} = \Gamma_{\phi\psi}^{\phi} = \cot \psi, \Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$$

We have

$$\Gamma_{\theta\theta}^{\psi} = -\sin \psi \cos \psi, \Gamma_{\phi\phi}^{\psi} = -\sin \psi \cos \psi \sin^2 \theta$$

$$\Gamma_{\psi\theta}^{\theta} = \Gamma_{\theta\psi}^{\theta} = \cot \psi \quad , \quad \Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta$$

$$\Gamma_{\psi\phi}^{\phi} = \Gamma_{\phi\psi}^{\phi} = \cot \psi, \Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$$

(b) 求 Riemannian tensor , Ricci tensor , Ricci scalar

$$R_{\sigma\mu\nu}^{\rho} = \partial_{\mu} \Gamma_{\nu\sigma}^{\rho} - \partial_{\nu} \Gamma_{\mu\sigma}^{\rho} + \Gamma_{\mu\lambda}^{\rho} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\rho} \Gamma_{\mu\sigma}^{\lambda}$$

$$R_{\psi\psi\theta}^{\psi} = \sin^2 \psi \quad , \quad R_{\theta\theta\psi}^{\psi} = -\sin^2 \psi \quad , \quad R_{\phi\psi\phi}^{\psi} = \sin^2 \psi \sin^2 \theta \quad , \quad R_{\phi\phi\psi}^{\psi} = -\sin^2 \psi \sin^2 \theta$$

$$R_{\psi\psi\theta}^{\theta} = -1 \quad , \quad R_{\psi\theta\psi}^{\theta} = 1 \quad , \quad R_{\phi\theta\phi}^{\theta} = \sin^2 \psi \sin^2 \theta \quad , \quad R_{\phi\phi\theta}^{\theta} = -\sin^2 \psi \sin^2 \theta$$

$$R_{\psi\psi\phi}^{\phi} = -1 \quad , \quad R_{\psi\phi\psi}^{\phi} = 1 \quad , \quad R_{\theta\theta\phi}^{\phi} = -\sin^2 \psi \quad , \quad R_{\theta\phi\theta}^{\phi} = \sin^2 \psi$$

例如

$$R_{\phi\psi\phi}^{\psi} = \partial_{\psi} \Gamma_{\phi\phi}^{\psi} - \partial_{\phi} \Gamma_{\psi\phi}^{\psi} + \Gamma_{\psi\lambda}^{\psi} \Gamma_{\phi\phi}^{\lambda} - \Gamma_{\phi\lambda}^{\psi} \Gamma_{\psi\phi}^{\lambda}$$

$$= \partial_{\psi} (-\sin \psi \cos \psi \sin^2 \theta) - \partial_{\phi} 0 + 0 - (-\sin \psi \cos \psi \sin^2 \theta)(\cot \psi)$$

$$= -\cos^2 \psi \sin^2 \theta + \sin^2 \psi \sin^2 \theta \cot^2 \psi = \sin^2 \psi \sin^2 \theta$$

$$R_{\psi\theta\psi}^{\theta} = \partial_{\theta}\Gamma_{\psi\psi}^{\theta} - \partial_{\psi}\Gamma_{\psi\theta}^{\theta} + \Gamma_{\theta\lambda}^{\theta}\Gamma_{\psi\psi}^{\lambda} - \Gamma_{\psi\lambda}^{\theta}\Gamma_{\psi\theta}^{\lambda}$$

$$= -\cancel{\partial}(\cancel{c}\cancel{\alpha}\cancel{\theta}) - (\cancel{\psi}\cancel{o}\cancel{t})\cancel{\lambda} = \csc^2\psi - \cot^2\psi = 1$$

$$R_{\theta\theta\phi}^{\phi} = \partial_{\theta}\Gamma_{\theta\phi}^{\phi} - \partial_{\phi}\Gamma_{\theta\theta}^{\phi} + \Gamma_{\theta\lambda}^{\phi}\Gamma_{\theta\phi}^{\lambda} - \Gamma_{\phi\lambda}^{\phi}\Gamma_{\theta\theta}^{\lambda}$$

$$= -\cancel{c}\cancel{s}\cancel{\partial}\cancel{\theta} + \cancel{c}\cancel{o}\cancel{\partial}\cancel{\theta} - \cancel{\alpha}\cancel{\phi}\cancel{o}\cancel{t} - (\cancel{\psi}\cancel{s}\cancel{i}\cancel{n}\cancel{\psi}\cancel{c}\cancel{\theta}\cancel{s})^2\cancel{\psi} \in \mathbb{O}^3$$

$$R_{\psi\psi} = R_{\psi\lambda\psi}^{\lambda} = 1+1=2 \quad , \quad R_{\theta\theta} = R_{\theta\lambda\theta}^{\lambda} = \sin^2\psi + \sin^2\psi = 2\sin^2\psi$$

$$R_{\phi\phi} = R_{\phi\lambda\phi}^{\lambda} = \sin^2\psi \sin^2\theta + \sin^2\psi \sin^2\theta = 2\sin^2\psi \sin^2\theta$$

The Ricci tensor is twice the metric $R_{\mu\nu} = 2g_{\mu\nu}$

The Ricci scalar $R = g^{\mu\mu}R_{\mu\mu} = 6$

§ 03 $I \times S^2$



一直線與一圓的乘積 $I \times S$ 是圓柱。

§ From $I \times S^2$ to S^3

The unit $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ with the Riemannian metric

induced from $\mathbb{R}^4$. Write $x^2 + y^2 + z^2 = 1 - w^2$, and use the spherical coordinates for

the slices of radius $r = \sqrt{1 - w^2}$

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \\ w = \sqrt{1 - r^2} \end{cases}$$

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad \text{and} \quad dw = \frac{-r}{\sqrt{1 - r^2}} dr, \quad dw^2 = \frac{r^2}{1 - r^2} dr^2$$

因此 $dx^2 + dy^2 + dz^2 + dw^2 = \frac{1}{1-r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$

The induced metric of S^3 from R^4 is $g = \frac{1}{1-r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$

From $I \times S^2$, $A(r) = (1-r^2)^{-\frac{1}{2}}$, $M = I \times S^2$ is locally isometric to S^3

$$R_{11} = \Omega_1^2(E_2, E_1) + \Omega_1^3(E_3, E_1) = \frac{A'}{rA^3} + \frac{A'}{rA^3} = \frac{2A'}{rA^3} = \dots = 2 \text{ 直接計算}$$

$$R_{22} = \Omega_2^1(E_1, E_2) + \Omega_2^3(E_3, E_2) = \frac{A'}{rA^3} + \frac{1}{r^2} \left(1 - \frac{1}{A^2}\right) = \dots = 1 + 1 = 2 \text{ 直接計算}$$

$$R_{33} = \Omega_3^1(E_1, E_3) + \Omega_3^2(E_2, E_3) = \frac{A'}{rA^3} + \frac{1}{r^2} \left(1 - \frac{1}{A^2}\right) = \dots = 2$$

$$R_{ij} = 2\delta_{ij} \text{ in an orthonormal frame.}$$

§ [RG01] p.81 有一個有意思的習作

$$S^3 = \{(x, y, z, w) \in R^4 \mid x^2 + y^2 + z^2 + w^2 = 2\}$$

$$T^2 = \{(x, y, z, w) \in R^4 \mid x^2 + y^2 = z^2 + w^2 = 1\}$$

The submanifold $T^2 \subset S^3$ splits S^3 into two connected components.

Let M be one of these components and let $\omega = zdx \wedge dy \wedge dw - xdy \wedge dz \wedge dw$ to be a 3-form.

Compute the two possible value of $\int_M \omega$

Stokes theorem

Let M be an n -dimensional oriented smooth manifold with boundary, let ω be a $(n-1)$ -differential form on M with compact support, and let $i: \partial M \rightarrow M$ be the inclusion of the boundary ∂M in M .

Then $\int_{\partial M} i^* \omega = \int_M d\omega$, where we consider ∂M with the

induced orientation.

T^2 divides S^3 into two connected components

$$M_1 := \{(x, y, z, w) \in S^3 \mid x^2 + y^2 < 1\} \text{ and } M_2 := \{(x, y, z, w) \in S^3 \mid x^2 + y^2 > 1\}$$

Let $\varphi_i: U_i \rightarrow M_i (i=1, 2)$ defined by

$$\varphi_i(r, v, v) = (r \cos v, r \sin v, \sqrt{2-r^2} \cos v, \sqrt{2-r^2} \sin v) \text{ on}$$

$$U_1 := (0, 1) \times (0, 2\pi) \times (0, 2\pi) \text{ and } U_2 := (1, \sqrt{2}) \times (0, 2\pi) \times (0, 2\pi). \text{ Then}$$

$$\varphi_i^* dx = \cos v dr - r \sin v dv$$

$$\varphi_i^* dy = \sin v dr + r \cos v dv$$

$$\varphi_i^* dz = -\frac{r}{\sqrt{2-r^2}} \cos v dr - \sqrt{2-r^2} \sin v dv$$

$$\varphi_i^* dw = -\frac{r}{\sqrt{2-r^2}} \sin v dr + \sqrt{2-r^2} \cos v dv$$

$$\begin{aligned} \text{And so } \varphi_i^* \omega &= (\sqrt{2-r^2} \cos v)(\varphi_i^* dx \wedge \varphi_i^* dy \wedge \varphi_i^* dw) - (r \cos v)(\varphi_i^* dy \wedge \varphi_i^* dz \wedge \varphi_i^* dw) \\ &= (r(2-r^2) \cos^2 v - r^3 \cos^2 v) dr \wedge dv \wedge dv \\ &\dots \end{aligned}$$

$$M \text{ 的參數 } x = r \cos \theta, y = r \sin \theta, z = \sqrt{2-r^2} \cos \phi, w = \sqrt{2-r^2} \sin \phi$$

...

$$\omega = [r(2-r^2) \cos^2 \phi - r^3 \cos \theta] dr \wedge d\theta \wedge d\phi$$

$$\int_0^{2\pi} \cos^2 \phi d\phi = \pi, \int_0^{2\pi} d\theta = 2\pi$$

$$\int_M \omega = 2\pi^2 \int_0^1 (2r - 2r^3) dr = \pi^2$$

$$\S \text{ 特殊酉群 } SU(2) = \left\{ \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix} \middle| a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

我們可以透過四元數來看到 $SU(2)$ 與 S^3 是同胚的。

四元數 $q = x_0 + x_1 i + x_2 j + x_3 k$ ， $|q|=1$ 形成一個群，這恰好就是 S^3 ，

而四元數的矩陣表示恰好對應於 $SU(2)$ ，因此， S^3 具有李群結構，並且與 $SU(2)$ 拓撲同構。

在李群 $SU(2)$ 上，我們可以考慮左不變的(left-invariant)結構。特別是左不變向量場(Lie 代數生成元)。

$SU(2)$ 的李代數 $\mathfrak{su}(2)$ 由所有跡為零的反厄米 2×2 矩陣組成，其標準基底為

$$T_1 = \frac{i}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, T_2 = \frac{i}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, T_3 = \frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

滿足 $\mathfrak{su}(2)$ 的交換關係： $[T_1, T_2] = T_3, [T_2, T_3] = T_1, [T_3, T_1] = T_2$

這與李代數 $\mathfrak{so}(3)$ 的交換關係相同，表明 $SU(2)$ 是 $SO(3)$ 的二重覆蓋。

在 $SU(2)$ 上，我們可以透過左乘作用推廣這些基底來定義左不變向量場。

設 $g \in SU(2)$ ，則左乘 $L_g : h \rightarrow gh$ 推動李代數 $\mathfrak{su}(2)$ 的基底，形成左不變向量場。

若改用四元數表示，則左不變向量場可寫為：

$$X_1 = x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} + x_0 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_0}$$

$$X_2 = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} + x_0 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_0}$$

$$X_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + x_0 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_0}$$

這些向量場對應於 $\mathfrak{su}(2)$ 的基底，並在 S^3 上生成左不變結構。

應用與幾何意義

1. 黎曼幾何

S^3 配上標準的度量，成為均勻空間，並且左不變向量場對應於 Killing 向量場，表示球面的對稱性。

2. 物理應用

$SU(2)$ 出現在量子力學與粒子物理中，特別是自旋 $\frac{1}{2}$ 系統與弱交互作用。

3. 聯絡與纖維叢

在愛因斯坦時空理論中， S^3 經常作為 $SO(4)$ 的子群出現，並與 Hopf 纖維結構相關。這些結構使得 S^3 與 $SU(2)$ 在數學與物理中有豐富的應用。

§ Show that the standard metric on $S^3 \cong SU(2)$ is left-invariant

Answer by ChatGPT :

A metric g on a Lie group G is left-invariant if left translation $L_g : G \rightarrow G$, given by

$L_g(h) = gh$ preserves the metric.

The left multiplication by elements of $SU(2)$ corresponds to isometries of S^3 in its natural Riemannian structure.

Verification:

1. The Euclidean metric in $\mathbb{R}^4$ is invariant under the left multiplication by $SU(2)$ elements, as this corresponds to an orthogonal action.
2. Thus, the induced metric on S^3 remains unchanged under left multiplication.

Since the standard metric on S^3 is induced from the Euclidean metric and left multiplication in $SU(2)$ acts as an isometry, the metric is left-invariant.

Answer by DeepSeek :

1. Isomorphism between $SU(2)$ and S^3 :

- Any element of $SU(2)$ can be written as a matrix $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$ where $|a|^2 + |b|^2 = 1$.
- By parameterizing a and b as $a = w + iz$ and $b = x + iy$, we get $w^2 + x^2 + y^2 + z^2 = 1$, which corresponds to the equation of S^3 .

2. Left-Invariance of the Metric:

- The standard metric on S^3 is induced from the Euclidean metric on $\mathbb{R}^4$.
- Left multiplication by an element of $SU(2)$ corresponds to quaternionic multiplication, which is an orthogonal transformation on $\mathbb{R}^4$.
- Orthogonal transformations preserve the Euclidean inner product, hence the induced metric on S^3 is preserved under left multiplication.
- The differential of left multiplication, being an orthogonal transformation, preserves the inner product on the tangent spaces, confirming the metric is left-invariant.

3. Conclusion:

- The identification of $SU(2)$ with S^3 and the fact that left translations are isometries (due to orthogonal transformations) show that the standard metric on S^3 is left-invariant under the group operation of $SU(2)$.

Thus , the standard metric on S^3 is left-invariant when S^3 is identified with the group $SU(2)$, as required by the group structure of $SU(2)$.

§ 隨波逐流的 S^3

For S^n of radius $r(t)$, the metric is given $g = r^2 \bar{g}$, where $\bar{g}$ is the metric on the unit sphere . The sectional curvature are all $\frac{1}{r^2}$.

The metric of unit 3-sphere , $\bar{g} = ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$

And of 3-sphere with radius $r=r(t)$, $g = ds^2 = r^2 d\psi^2 + r^2 \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$

$$g = r^2 \bar{g}$$

For a n -sphere , $\text{Ric}(g) = (n-1)g$, so the Ricci flow equation becomes a ODE .

$$\frac{\partial g}{\partial t} = -2\text{Ric}(g) \Rightarrow \frac{\partial}{\partial t}(r^2 \bar{g}) = -2(n-1)\bar{g} \Rightarrow \frac{dr^2}{dt} = -2(n-1)$$

$$r^2 = R_0^2 - 2(n-1)t$$

$r(t) = \sqrt{R_0^2 - 2(n-1)t}$, as $t \rightarrow \frac{R_0^2}{2(n-1)}$, the sphere shrinks to a point (a singularity) .

Where $n=3$

Similarly , for hyperbolic n -space $H^n (n > 1)$, the Ricci flow reduces to the ODE

$$\frac{d(r^2)}{dt} = 2(n-1) \text{ which has the solution } r(t) = \sqrt{R_0^2 + 2(n-1)t}$$

So the solution expands out to infinity .

§ 06 參考資料

1. [Spacetime and Geometry] Ch3 EX08 and EX16

2. $I \times S^2$, $g = A^2(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$

S^2 做為一個 Riemannian manifold $g = d\theta^2 + \sin^2 \theta d\phi^2$ induced from R^3

3. [Wormhole metric](#) by Ellis

$$c^2 dt^2 = d\rho^2 + (\rho^2 + n^2)(d\theta^2 + \sin^2 \theta d\phi^2) \text{ where } n \text{ is the drainhole parameter}$$

4. $ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2 \theta d\phi^2)$ by MT wormhole

5. Geometry of 3-Sphere [Garret Sobczyk](#) [What is a Pauli matrix]

Ex

Consider a smooth map $f : S^3 \rightarrow S^2$

(a) Let α be a 2-form on S^2 such that $\int_{S^2} \alpha = 1$.

Show that there exists a 1-form η on S^3 such that $f^* \alpha = d\eta$

(b) Show that the value of the integral $\int_{S^3} \eta \wedge d\eta$ is independent of the choices of α

and η . (Hence it depends only on f , and is called the Hopf invariant of f .)

(c) Show that the Hopf invariant of f depends only on the homotopy class of f .