Given the fact that every vector field on S^2 must vanish somewhere(the Hairy Ball theorem)，show that S^2 has no Lie group structure。

1. Parallelizability：Lie groups are parallelizable，meaning their tangent bundles are trivial。This is achieved by left-translating a basis of the tangent space at the identity to all points。However, S^2 is not parallelizable due to the Hairy Ball Theorem，which prevents the existence of a global non-vanishing vector field。Hence，S^2 cannot be a Lie group。
2. Euler Characteristic：The Euler characteristic $\chi(S^2)=2$。

   For compact Lie groups，the Euler characteristic must be zero (as they admit non-vanishing vector fields，leading to a zero index sum via the Poincaré-Hopf theorem)。 The non-zero Euler characteristic of S^2 directly contradicts this requirement。

Conclusion：Since S^2 is neither parallelizable nor has Euler characteristic zero，it cannot admit a Lie group structure。

若S^2是一個 Lie group，那麼它同時是一個流形與群，並且滿足：

1. 它的單位元 e 處的切空間T_eS^2形成一個李代數$\mathfrak{g}$。
2. 透過左平移（left translation），李代數中的每個向量都可以擴展成整個S^2上的左不變向量場。
3. 關鍵事實：在 Lie 群上，左不變向量場不可能在任何點上消失，因為它們來自李代數中的基底。

矛盾。因此S^2不可能成為一個 Lie 群。