§ S^2

[An Introduction to Riemannian Geometry **Jose Natario**] Ex 3.3 (4)

$\varphi : M \to N$ is an immersion。(N, g) is a Riemannian manifold，then $\varphi^* g$ is a Riemannian metric in M induced by φ。

$\phi : S^2 \to R^3$

$(0,\pi)\times(0,2\pi)\xrightarrow{\phi}\mathrm{R}^3$

$\phi(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

(a) Using these coordinates，determine the expression of the Riemnnian metric induced on S^2 by the Euclidean metric of R^3。

$(R^3, g), g = dx^2 + dy^2 + dz^2$，$(S^2, \tilde{g}), \tilde{g} = \varphi^* g$

$$\frac{\partial}{\partial\theta} = \frac{\partial\phi}{\partial\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\frac{\partial}{\partial\varphi} = \frac{\partial\phi}{\partial\varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

$$\tilde{g}_{11} = g_{\theta\theta} = <\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}> = 1，$$

$$\tilde{g}_{12} = \tilde{g}_{21} = g_{\theta\varphi} = g_{\varphi\theta} = <\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}> = 0，\tilde{g}_{22} = g_{\varphi\varphi} = <\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}> = \sin^2\theta$$

$$\therefore \varphi^* g = d\theta^2 + \sin^2\theta d\varphi^2$$

Or，on S^2 $x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta$

$dx = \cos\theta\cos\varphi dx - \sin\theta\sin\varphi d\varphi$

$dy = \cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi$

$dz = -\sin\theta d\theta$

Then $dx^2 + dy^2 + dz^2 = \ldots = d\theta^2 + \sin^2\theta d\varphi^2$

$\phi^* g = d\theta^2 + \sin^2\theta d\varphi^2$ is the induced metric on S^2

(b) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates

$$\left(g_{ij}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}，\left(g^{ij}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sin^2\theta} \end{pmatrix}$$

$$\Gamma^{\theta}_{\varphi\varphi}=\frac{1}{2}\sum_{l=1}^{2}g^{\theta l}\left(\frac{\partial g_{\varphi l}}{\partial\varphi}+\frac{\partial g_{\varphi l}}{\partial\varphi}-\frac{\partial g_{\varphi\varphi}}{\partial x^{l}}\right)=\ldots=\frac{1}{2}\left(-\frac{\partial\sin^{2}\theta}{\partial\theta}\right)=-\sin\theta\cos\theta$$

同理 $\Gamma^{\varphi}_{\theta\varphi}=\Gamma^{\varphi}_{\varphi\theta}=\cot\theta$

Or let $L=(\dot{\theta})^{2}+\sin^{2}\theta(\dot{\phi})^{2}$ ，consider the Euler equation

$$\frac{\partial L}{\partial\theta}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=0\Rightarrow 2\sin\theta\cos\theta(\dot{\phi})^{2}-\frac{d}{d\tau}(2\dot{\theta})=0$$

$\ddot{\theta}-\sin\theta\cos\theta(\dot{\phi})^{2}=0$ imply $\Gamma^{\theta}_{\phi\phi}=-\sin\theta\cos\theta$

$$\frac{\partial L}{\partial\phi}-\frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{\phi}}\right)=0\Rightarrow\frac{d}{d\tau}(\sin^{2}\theta\times(2\dot{\phi}))=0$$

$2\sin\theta\cos\theta\dot{\theta}\dot{\phi}+\sin^{2}\theta\ddot{\phi}=0$ $\ddot{\phi}+2\cot\theta\dot{\theta}\dot{\phi}=0$ imply $\Gamma^{\phi}_{\theta\phi}=\Gamma^{\phi}_{\phi\theta}=\cot\theta$

(c) Show that the equator is the image of a geodesic。

Geodesic equation $\ddot{x}^{i}+\sum_{j,k}\Gamma^{i}_{jk}\dot{x}^{j}\cdot\dot{x}^{k}=0$ for i=1，2

得 $\ddot{\theta}-\sin\theta\cos\theta(\dot{\phi})^{2}=0$ ，$\ddot{\phi}+2\cot\theta\dot{\theta}\dot{\phi}=0$

赤道 $\begin{cases}\theta(t)=\dfrac{\pi}{2}\\ \phi(t)=t\end{cases}$

(d) Show that any rotation about an axis through the origin in R^3 induces an isometry of S^2

Any rotation about an axis through the origin in R^3 is an isometry of R^3 which preserves S^2。

Since we are considering the metric in S^2 induced by the Euclidean metric on R^3，it is clear that such a rotation will determine an isometry of S^2。

(isometry 等距同構)

(e) Show that the images of geodesics of S^2 are great circles

Given a point $p \in S^2$ and a vector $v \in T_pS^2$, there exists a rotation $R : \mathbb{R}^3 \to \mathbb{R}^3$ such that $R(p) = (1,0,0)$ and $R(v) = (0,1,0)$. The geodesic with these initial conditions is clearly the curve c given in coordinates by $\hat{c}(t) = (\theta(t), \varphi(t)) = \left(\frac{\pi}{2}, t\right)$, whose image is the equator. By Exercise 3.3(3), the geodesic with initial condition $v \in T_pS^2$ must be $R^{-1} \circ c$. Since the image of c is the intersection of S^2 with the plane $z = 0$, the image of $R^{-1} \circ c$ is the intersection of S^2 with some plane through the origin, i.e. a great circle.

(f) Find a geodesic triangle whose internal angles add up to $\dfrac{3\pi}{2}$

For example the triangle with vertices (1,0,0)，(0,1,0)，(0,0,1)

(g) Let $c: R \to S^2$ be given by $c(t) = (\sin\theta_0 \cos t, \sin\theta_0 \sin t, \cos\theta_0)$，where

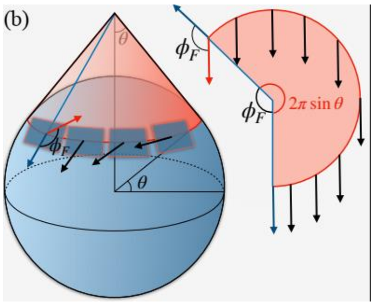
$\theta_0 \in (0, \frac{\pi}{2})$ (therefore c is not a geodesic)。

Let V be a vector field parallel along c such that $V(0) = \frac{\partial}{\partial\theta}$ ($\frac{\partial}{\partial\theta}$ is well defined at

$(\sin\theta_0, \cos\theta_0)$ by continuity)。

Compute the angle by which V is rotated when it returns to the initial point。

[Foucault pendupum]



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[From the geometry of Foucault pendulum to the topology of planetary waves]

The metric on S^2 is $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$，

$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$

一個 vector field V^μ 沿曲線 $x^\mu(\lambda)$ 平行移動則 $\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{ij}\frac{dx^i}{d\lambda}V^j = 0$

Along the curve c， $\theta = \theta_0$ is a constant， $\phi = t$ then $\dot{\theta} = 0, \dot{\phi} = \frac{d\phi}{dt} = 1$

The equations for paralleltransport are $\dot{V}^i+\sum_{j,k}\Gamma^i_{jk}\dot{x}^jV^k=0$

$\dot{V}^\theta+\sum_{ij}\Gamma^\theta_{ij}\dot{x}^iV^j=0$ ， $\dot{V}^\theta+\Gamma^\theta_{\phi\phi}\dot{\phi}V^\phi=0$ ， $\dot{V}^\theta-\sin\theta_0\cos\theta_0V^\phi=0\ldots(1)$

$\dot{V}^\varphi+\sum_{i,j}\Gamma^\varphi_{ij}\dot{x}^iV^j=0$ ， $\dot{V}^\phi+\Gamma^\phi_{\theta\phi}\dot{\theta}V^\phi+\Gamma^\phi_{\phi\theta}\dot{\phi}V^\theta=0$ ， $\dot{V}^\phi+\cot\theta_0V^\theta=0\ldots(2)$

Since $\dot{\theta}=0,\dot{\phi}=1$

(1)式兩邊對 t 微分，$\ddot{V}^\theta=\sin\theta_0\cos\theta_0\dot{V}^\phi=-\sin\theta_0\cos\theta_0\cot\theta_0V^\theta=-\cos^2\theta_0V^\theta$

解此微分方程，$V^\theta=A\cos(t\cos\theta_0)+B\sin(t\cos\theta_0)$

$$V^\phi=\frac{1}{\sin\theta_0\cos\theta_0}\dot{V}^\theta=\frac{1}{\sin\theta_0\cos\theta_0}(-\cos\theta_0A\sin(t\cos\theta_0)+\cos\theta_0B\cos(t\cos\theta_0))$$

$$V(0)=\frac{\partial}{\partial\theta}=(V^\theta(0),V^\phi(0))=(1,0)\Rightarrow V^\theta(0)=1,V^\phi(0)=0$$

Imply A=1、B=0，so $V^\theta=\cos(t\cos\theta_0),V^\phi=-\frac{1}{\sin\theta_0}\sin(t\cos\theta_0)$

Note that $|V|=g_{\mu\nu}V^\mu V^\nu=V^\theta V^\theta+\sin^2\theta_0V^\phi V^\phi=1$

假設$V(0),V(2\pi)$的夾角為α ，

$$V(0)=\frac{\partial}{\partial\theta},V(2\pi)=\cos(2\pi(\cos\theta_0))\frac{\partial}{\partial\theta}-\frac{1}{\sin\theta_0}\sin(2\pi(\cos\theta_0))\frac{\partial}{\partial\phi}$$

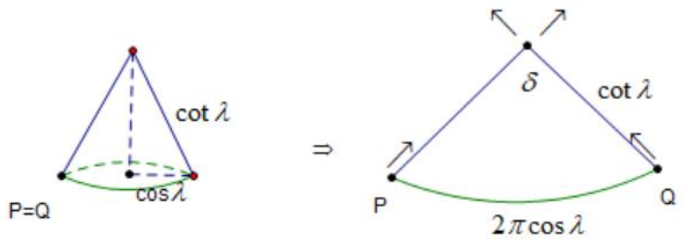
則$\cos\alpha=<V(0),V(2\pi)>=\cos(2\pi\cos\theta_0)$

所以$\alpha=2\pi\cos\theta_0$或$2\pi(1-\cos\theta_0)$

若傅科擺位於北緯λ度($\lambda=\frac{\pi}{2}-\theta_0$)， 則擺動方向每 24 小時轉動$2\pi\sin\lambda$

參考 [大域微分幾何]p.156~162

Do Carmo 第二章習作 p.58



$2\pi\cos\lambda=\cot\lambda\times\delta\ \Rightarrow\ \delta=2\pi\sin\lambda$

(h) Use this result to prove that no open set $U \subset S^2$ is isometric to an open set $W \subset R^2$ with the Euclidean metric

Using the fact that any point on S^2 can be carried to $(0, 0, 1)$ by an appropriate isometry, we just have to show that no open neighborhood $U \subset S^2$ of $(1, 0, 0)$ is isometric to an open set $V \subset \mathbb{R}^2$ with the Euclidean metric. Now any such neighborhood contains the image of a curve $c(t)$ as given in (g) (for $\theta_0 > 0$ sufficiently small). If U were isometric to W, the Levi–Civita connection on U would be the trivial connection, and hence the parallel vector field $V(t)$ in (g) would satisfy $V(0) = V(2\pi)$. Since this is not true for any $\theta_0 \in (0, \frac{\pi}{2})$, U cannot be isometric to W.

(i) Given a geodesic $c : \mathbb{R} \to \mathbb{R}^2$ of $\mathbb{R}^2$ with the Euclidean metric and a point $p \notin c(\mathbb{R})$, there exists a unique geodesic $\tilde{c} : \mathbb{R} \to \mathbb{R}^2$ (up to reparameterization) such that $p \in \tilde{c}(\mathbb{R})$ and $c(\mathbb{R}) \cap \tilde{c}(\mathbb{R}) = \varnothing$ (**parallel postulate**). Is this true in S^2?

(i) 求S^2的 Gauss curvature

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \text{ , } \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

$$R^{\theta}_{\phi\theta\phi} = \partial_\theta\Gamma^{\theta}_{\phi\phi} - \partial_\phi\Gamma^{\theta}_{\theta\phi} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\theta\phi}$$

$$= (\sin^2\theta - \cos^2\theta) + 0 - 0 - (-\sin\theta\cos\theta)(\cot\theta)$$

$$R_{\theta\phi\theta\phi} = g_{\theta\lambda}R^{\lambda}_{\phi\theta\phi} = g_{\theta\theta}R^{\theta}_{\phi\theta\phi} = a^2\sin^2\theta$$

All the components of the Riemann tensor either vanish or are related to this one by symmetry。

The Ricci tensor $R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu}$

$$R_{\theta\theta} = g^{\phi\phi}R_{\phi\theta\phi\theta} = 1 \text{ , } R_{\theta\phi} = R_{\phi\theta} = 0 \text{ , } R_{\phi\phi} = g^{\theta\theta}R_{\theta\phi\theta\phi} = \sin^2\theta$$

The Ricci scalar $R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \dfrac{2}{a^2}$