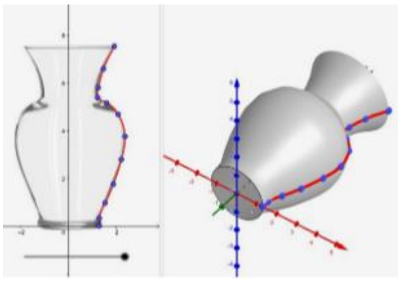

$\Gamma^i_{jk}$ 的算法有兩種。

§ $R^3$ 中的旋轉面， $x = r\cos\theta, y = r\sin\theta, z = f(r)$ ，f(r)可任意微分。

(1) $ds^2 = Edr^2 + 2Fdrd\theta + Gd\theta^2$ ，求 E，F，G

(2) 求 $\Gamma^k_{ij}$

(3) 寫下測地線方程式

(4) 解測地線方程式

(5) $\theta = c$ (常數)是否測地線？為什麼？

(6) 若 $\dfrac{dz}{dr} = \dfrac{\sqrt{b^2 - r^2}}{r}$ (b 是常數)，求高斯曲率

(1) $\mathrm{X}(r,\theta) = (r\cos\theta, r\sin\theta, f(r))$

$X_r = (\cos\theta, \sin\theta, f'(r)) \quad X_\theta = (-r\sin\theta, r\cos\theta, 0)$

$E = 1 + (f'(r))^2, F = 0, G = r^2$

$ds^2 = (1 + (f'(r)^2)dr^2 + r^2 d\theta^2$

(2) $\Gamma^i_{jk} = \dfrac{1}{2} g^{il} (\dfrac{\partial g_{kl}}{\partial x^j} + \dfrac{\partial g_{jl}}{\partial x^k} - \dfrac{\partial g_{jk}}{\partial x^l})$

$$(g_{ij}) = \begin{pmatrix} 1+f'^2 & 0 \\ 0 & r^2 \end{pmatrix}, (g^{ij}) = \begin{pmatrix} \dfrac{1}{1+f'^2} & 0 \\ 0 & \dfrac{1}{r^2} \end{pmatrix}$$

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr} \left\{ \frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial r} \right\} = \frac{f'f''}{1+f'^2}$$

$\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \Gamma^\theta_{rr} = 0$ ， $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \dfrac{1}{r}$ ， $\Gamma^\theta_{\theta\theta} = 0$ ， $\Gamma^r_{\theta\theta} = \dfrac{-r}{1+f'^2}$

$$\left(\Gamma^r_{jk}\right) = \begin{pmatrix} \dfrac{f'f''}{1+f'^2} & 0 \\ 0 & \dfrac{-r}{1+f'^2} \end{pmatrix}, \left(\Gamma^\theta_{jk}\right) = \begin{pmatrix} 0 & \dfrac{1}{r} \\ \dfrac{1}{r} & 0 \end{pmatrix}$$

$\Gamma^i_{jk}$ 的另一種算法是：

The Lagrangian $L = (1+f'^2)\dot{r}^2 + r^2\dot{\theta}^2$

The E-L equation $\frac{d}{ds}(\frac{\partial L}{\partial \dot{\theta}})-(\frac{\partial L}{\partial \theta})=0 \quad \Rightarrow \frac{d}{ds}(2r^2\dot{\theta})=0 \Rightarrow r^2\dot{\theta}=c$

(3) Geodesic equation： $\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^k=0$

$$\ddot{r}+\frac{f'f''}{1+f'^2}(\dot{r})^2-\frac{r}{1+f'^2}(\dot{\theta})^2=0 \quad ...(*)$$

$\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}=0 ...(**) \Rightarrow \frac{1}{r^2}\frac{d}{ds}(r^2\frac{d\theta}{ds})=0 \Rightarrow r^2\frac{d\theta}{ds}=c$ 與上面用 Lagrangian 算出來的相同！

And from (*) (**) ， $(\Gamma^r_{jk})=\begin{pmatrix}\frac{f'f''}{1+f'^2} & 0 \\ 0 & \frac{-r}{1+f'^2}\end{pmatrix}$ and $(\Gamma^\theta_{jk})=\begin{pmatrix}0 & \frac{1}{r} \\ \frac{1}{r} & 0\end{pmatrix}$ followed。

$ds=\frac{r^2}{c}d\theta$ 代入 $ds^2=(1+f'^2)dr^2+r^2d\theta^2$

$$d\theta=\pm c\cdot\frac{1+f'^2}{r\sqrt{r^2-c^2}}dr$$

$$\theta-\theta_0=\pm c\int_{r_0}^{r}\frac{\sqrt{1+f'(\xi)^2}}{\xi\sqrt{\xi^2-c^2}}d\xi$$

If c=0， $\theta=\theta_0$ is a constant，imply that meridian(經線) are geodesics。

(or for $\theta=\theta_0$ then $\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}=0$ will obviously holds。)

If $c=0$, then the first equation above gives $\phi$ =constant, so the meridians are geodesics. The parallels $r$ =constant are geodesics when $f'(r)=\infty$ in which case the tangent bundle restricted to the parallel is a cylinder with a vertical generator.

In the particular case of a cone of revolution with a generator that makes an angle $\alpha$ with the $z$-axis, $f(r)=\cot(\alpha)r$, equation 6.58 becomes:

$$\phi=\pm c\int\frac{\sqrt{1+\cot^2\alpha}}{r\sqrt{r^2-c^2}}\,dr$$

which can be immediately integrated to yield:

$$\phi=\pm\csc\alpha\sec^{-1}(r/c) \qquad (6.59)$$

where $c$ is a constant of integration. If the geodesic $\alpha(s)=\alpha(r(s),\phi(s))$ represents the path of a free particle constrained to move in the surface, this conserved quantity is essentially the angular momentum. A neat result can be obtained by considering the angle $\sigma$ that the tangent vector $V=\alpha'$ makes with a meridian. Recall that the length of $V$ along the geodesic is constant, so let's set $\|V\|=k$ From the chain rule we have

$$\alpha'(t)=\mathbf{x}_r\frac{dr}{ds}+\mathbf{x}_\phi\frac{d\phi}{ds}.$$

Then

$$\begin{aligned}\cos\sigma &= \frac{<\alpha',\mathbf{x}_\phi>}{\|\alpha'\|\cdot\|\mathbf{x}_\phi\|}=\frac{G\frac{d\phi}{ds}}{k\sqrt{G}},\\ &= \frac{1}{k}\sqrt{G}\frac{d\phi}{ds}=\frac{1}{k}r\dot{\phi}.\end{aligned}$$

We conclude from 6.56, that for a surface of revolution, the geodesics make an angle $\sigma$ with meridians that satisfies the equation

$$r\cos\sigma=\text{constant}. \tag{6.57}$$

This result is called Clairaut's relation。

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後面接下去是 MT Wormhole 的 geodesic p.159

對球面而言， $X(\theta,\varphi)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$ ， $r=\sin\theta, f(r)=\sqrt{1-r^2}$

$\mathrm{X}(r,\theta)=(r\cos\theta, r\sin\theta, f(r))$

r-curve(即$\theta$=constant)都是測地線。

$\theta$-curve(r(s)=constant)，for a curve $\alpha(s)$ ，(*)(**) imply that $\dot{\theta}=0$ ， $\theta(s)$ =constant

$\theta$-curve 稱為 parallel，所有的 parallel 都是圓，a parallel on a surface of revolution is not a geodesic。

Let $r=b\sin\theta$ then

$$\int\frac{\sqrt{b^2-r^2}}{r}dr=b\int\frac{\cos^2\theta}{\sin\theta}d\theta=b(-\ln\left|\csc\theta+\cot\theta\right|+\cos\theta)+c=b(-\ln\left|\frac{1+\sqrt{1-r^2}}{r}\right|+\sqrt{1-r^2})+c$$

$$\frac{dz}{dr}=\frac{\sqrt{b^2-r^2}}{r} \quad z=f(r)=b(-\ln\left|\frac{1+\sqrt{1-r^2}}{r}\right|+\sqrt{1-r^2})+c$$

$$f'(r)=\frac{dz}{dr}=\frac{\sqrt{b^2-r^2}}{r}, f''(r)=\frac{-b^2}{r^2\sqrt{b^2-r^2}}$$

$$\mathrm{X}(r,\theta) = (r\cos\theta, r\sin\theta, f(r))$$
$$X_r = (\cos\theta, \sin\theta, f'(r)) \quad X_\theta = (-r\sin\theta, r\cos\theta, 0)$$
$$E = 1 + (f'(r))^2, F = 0, G = r^2$$
$$X_{rr} = (0,0,f''(r)) \quad X_{r\theta} = (-\sin\theta, \cos\theta, 0) \quad X_{\theta\theta} = (-r\cos\theta, -r\sin\theta, 0)$$

$$N = \frac{X_r \times X_\theta}{\left|X_r \times X_\theta\right|} = \frac{(-f'(r)\cos\theta, -f'(r)\sin\theta, 1)}{\sqrt{1+(f'(r))^2}}$$

$$e = X_{rr} \bullet N = \frac{f''}{\sqrt{1+f'^2}} \quad f = 0 \quad g = X_{\theta\theta} \bullet N = \frac{rf'}{\sqrt{1+f'^2}}$$

$$K = \frac{eg - f^2}{EG - F^2} = \frac{rf'f''/(1+f'^2)}{r^2(1+f'^2)} = \frac{f'f''}{r}$$

Now $z = f(r), f'(r) = \frac{dz}{dr} = \frac{\sqrt{b^2 - r^2}}{r}, f''(r) = \frac{-b^2}{r^2\sqrt{b^2 - r^2}}$

$$K = \frac{-b^2}{r^4}$$