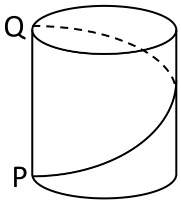
§ Geodesics

§ 01　Geodesic equation： $\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^k=0$

一曲面上連接兩點 有最短距離的曲線，稱為測地線(geodesic)。
這樣的觀念在高等數學很重要，在初等數學 我們找到一些例子。



左圖一隻螞蟻從一圓柱體的糖罐底部 P 點沿著柱體側面旋轉往上爬到 Q 點，若柱體高 $\overline{PQ}$=8 公分，底面圓周長 6 公分，求螞蟻爬行的最短距離___公分。
(剪開 攤開 10 公分)

這個例子也說明測地線不一定是最短路徑，螞蟻直接從 P 往上走到 Q，顯然距離比較近。

在相對論中，等價(效)原理(equivalence principle)指出重力場決定伽利略時空的曲線，這些曲線是對稱聯絡(Cartan connection)的測地線。
在廣義相對論中，一個自由粒子走的路徑是測地線。

An important property of geodesic in a spacetime with Lorentzian metric is that the character (timelike/null/spacelike)of the geodesic relative to a metric compatible connection，never changes。
This is simple because parallel transport preserves inner product，and the character is determined by the inner product of the tangent vector itself。

§ 02 Viewpoint of classical mechanism

In Lagrangian mechanism， $C=\left\{c\middle|c:[a,b]\to M, c(a)=p, c(b)=q\right\}$

A Lagrangian function L on M， $L:TM\to R$

The action determined by L on C， $S:C\to R$ ， $S(c):=\int_a^b L(\dot{c}(t))dt$

$\gamma:(-\varepsilon,\varepsilon)\to C$ a variation of c
such that $\gamma(0)=c$ and $\gamma(s)(t):(-\varepsilon,\varepsilon)\times[a,b]\to M$ is differentiable

$\tilde{\gamma}(s,t) := \gamma(s)(t)$ 。If $\frac{d}{ds}|_{s=0} S(\gamma(s)) = 0$ for $\forall \gamma \in C$ ，then curve c is called the critical point of the action S。(Least action principle)

Theorem

$c \in C$ is the critical point of S which determined by the Lagrangian L$\Leftrightarrow$ c satisfies Euler-Lagrange equations。

$\gamma : [a,b] \to M$ is a smooth curve on Riemann manifold

$L(\gamma) := \int_a^b \left| \frac{d\gamma}{dt}(t) \right| dt$ is the length of γ

$E(\gamma) := \frac{1}{2} \int_a^b \left| \frac{d\gamma}{dt}(t) \right|^2 dt = \frac{1}{2} \int_a^b (g_{ij} \dot{x}^i \dot{x}^j) dt$ is the energy of γ

(In physics, $E(\gamma)$ is usually called the "action of γ"，where γ is considered as the orbit of a mass point。)

Represent by coordinates $(x^1(\gamma(t)), x^2(\gamma(t)), \ldots, x^m(\gamma(t)))$

$\dot{x}^i(t) := \frac{d}{dt}(x^i(\gamma(t)))$ then

$L(\gamma) = \int_a^b \sqrt{g_{ij}(x(\gamma(t)) \dot{x}^i(t) \dot{x}^j(t)} dt$ ，$E(\gamma) = \frac{1}{2} \int_a^b g_{ij}(x(\gamma(t)) \dot{x}^i(t) \dot{x}^j(t) dt$

Then Euler-Lagrange equation of energy E is

$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t)) \dot{x}^j(t) \dot{x}^k(t) = 0$ ，I =1,2,3,…i.e. geodesic equation

Proof

The Euler-Lagrange equation of a functional $I(x) = \int_a^b f(t, x(t), \dot{x}(t)) dt$

Is given by $\frac{d}{dt} \frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0$

$E(\gamma) = \frac{1}{2} \int g_{jk}(x(t)) \dot{x}^j \dot{x}^k dt$ ($g_{ij} \dot{x}^i \dot{x}^j = L(t, x(t), \dot{x}(t))$ ，and the Euler-Lagrange equation is $\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = 0$)，then $\frac{d}{dt}(g_{ik} \dot{x}^k + g_{ji} \dot{x}^j) - g_{jk,l} \dot{x}^j \dot{x}^k = 0$

For i=1,2,…,m therefor

$$g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j + g_{ik,\ell}\dot{x}^\ell\dot{x}^k + g_{ji,\ell}\dot{x}^\ell\dot{x}^j - g_{jk,i}\dot{x}^j\dot{x}^k = 0.$$

Renaming some indices and using the symmetry $g_{ik} = g_{ki}$, we get

$$2g_{\ell m}\ddot{x}^m + (g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0, \quad \ell = 1,\dots,d, \qquad (1.4.16)$$

and from this

$$g^{i\ell}g_{\ell m}\ddot{x}^m + \frac{1}{2}g^{i\ell}(g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0, i = 1,\dots,d.$$

Because of

$$g^{i\ell}g_{\ell m} = \delta_{im}, \text{ and thus } g^{i\ell}g_{\ell m}\ddot{x}^m = \ddot{x}^i,$$

We get $\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0$

Q.E.D. [Geometric analysis] p.23

Thus，geodesics are the critical points of the energy function。

Geodesics are paths of minimal energy。

§ 03

$\dfrac{d^2u^i}{ds^2} + \Gamma^i_{jk}\dfrac{du^j}{ds}\dfrac{du^k}{ds} = 0 \cdots$ equation of a geodesic (1)

$\dot{V}^i + \sum_{j,k}\Gamma^i_{jk}\dot{x}^j V^k = 0$ Y parallel transported along X (2)

Covariant derivative of Y along X， $\nabla_X Y = \sum_i (XY^i + \sum_{j,k}\Gamma^i_{jk}X^jY^k)\dfrac{\partial}{\partial x^i}$ (3)

(1) (2) (3)是互相決定的。

§ 04 A geodesic is a curve along which the tangent vector is parallel transported。

$\nabla_{\frac{\partial}{\partial x^i}}\dfrac{\partial}{\partial x^j} := \Gamma^i_{jk}\dfrac{\partial}{\partial x^k}$ then

$$\nabla_{\frac{\partial}{\partial x^i}}(w^j\frac{\partial}{\partial x^j}) = w^j\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} + \frac{\partial w^j}{\partial x^i}\frac{\partial}{\partial x^j} = w^j\Gamma^i_{jk}\frac{\partial}{\partial x^k} + \frac{\partial w^j}{\partial x^i}\frac{\partial}{\partial x^j}$$

And $\nabla_{v^i\frac{\partial}{\partial x^i}}\dfrac{\partial}{\partial x^j} = v^i\nabla_{\frac{\partial}{\partial x^i}}\dfrac{\partial}{\partial x^j}$

For a curve x(t)，with $\dfrac{d}{dt} = \dfrac{dx^i}{dt}\dfrac{\partial}{\partial x^i}$

$\nabla_{\frac{d}{dt}}\dfrac{dx}{dt} = (\dfrac{d^2x^i}{dt^2} + \Gamma^i_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt})\dfrac{\partial}{\partial x^i}$ ， $\dfrac{d^2x^i}{dt^2} + \Gamma^i_{jk}\dfrac{dx^j}{dt}\dfrac{dx^k}{dt} = 0$ is the geodesic equation

That is $\nabla_X X = 0$

§ 05 covariant derivative 平行移動 協變微分

$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$

Parallel transport of the tensor T along the path $x^\mu(\lambda)$ ， $\frac{D}{d\lambda} T = 0$

$\frac{d}{d\lambda} V^\mu + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} V^\rho = 0$ for a vector V^μ

$\frac{D}{d\lambda} \frac{dx^\mu}{d\lambda} = 0$ or

$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$

§ 06 Minimal-coupling principle 最小耦合原理

就是說 粒子在不受力的情形下

(1)寫出平直(歐氏)空間的運動方程式(直線)

(2)把加速度中的微分改成協變微分

(3)得到彎曲空間中的運動方程式(即 測地線方程式)

$\frac{d^2 x^\mu}{d\lambda^2} = 0$ for a free particle

$\to \frac{dx^\nu}{d\lambda} \partial_\nu \frac{dx^\mu}{d\lambda} \to \frac{dx^\nu}{d\lambda} \nabla_\lambda \frac{dx^\mu}{d\lambda} = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda}$

§ 07 method of variation 變分法

$\delta \int_a^b \sqrt{g_{ij} \dot{x}^i \dot{x}^j} dt = 0$ Yves Talpaert p.30

[Differential Geometry with Applications to Mechanics and Physics]

Euler-Lagrange equation $\frac{d}{dt}(\frac{\partial F}{\partial \dot{x}^i}) - \frac{\partial F}{\partial x^i} = 0$ ， with $F = \sqrt{g_{ij} \dot{x}^i \dot{x}^j}$

We establish the geodesic equations from the calculus of variations。

On a manifold，a geodesic joining two points x_a, x_b is a curve such that

$$\delta\int_{s_a}^{s_b} ds = \delta\int_a^b \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}dt = 0$$

The calculus of variations means it is a curve “satisfying” the Euler

equation $\frac{d}{dt}(\frac{g_{ij}\dot{x}^j}{F}) - \frac{1}{2F}\partial_i(g_{jk})\dot{x}^j\dot{x}^k = 0$

Substituting the curvilinear parameter s for t，the geodesic equations are

$$\frac{d}{dt}(g_{ij}\frac{dx^j}{dt}) - \frac{1}{2ds}\partial_i(g_{jk})\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

$$g_{ij}\frac{d^2x^j}{ds^2} + \partial_k g_{ij}\frac{dx^k}{ds}\frac{dx^j}{ds} - \frac{1}{2}\partial_i g_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

$$g_{ij}\frac{d^2x^j}{ds^2} + \frac{1}{2}(\partial_k g_{ij}\frac{dx^k}{ds}\frac{dx^j}{ds} + \partial_j g_{ik}\frac{dx^j}{ds}\frac{dx^k}{ds}) - \frac{1}{2}\partial_i g_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

By multiplying by g^{ri}，we obtain the geodesic equations

$\frac{d^2x^r}{ds^2} + \Gamma^r_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$，where s is proportional to t。

看起來與§ 02 相同。

§ 08 heat flow method 從熱力學 heat flow 方程看測地線

Heat equation on R

(1) $u_t - ku_{xx} = 0$

(2) $\begin{cases} u_t - ku_{xx} = f(x,t), t > 0 \\ u\big|_{t=0} = g(x) \end{cases}$

Consider $u^i_t = u^i_{ss} + \Gamma^i_{jk}u^j_s u^k_s, u(s,0) = \gamma(s)$ for $s \in S^1$ then

1. $u^i_t - u^i_{ss} = f$ is a heat equation
2. $u^i_{ss} + \Gamma^i_{jk}u^j_s u^k_s = 0$ is a geodesic equation

c.f.

1. RG4701 中有 Hyperbolic plane 的例子。有一個解微分方程的巧妙的方法。
2. RG3303 是 geodesic flows
3. RG5101 是 Jacobi 場一個 Jacobi field J(t)即沿著 geodesic γ 滿足 Jacobi equation 的向量場。$\frac{D^2J}{dt^2}+R(\gamma'(t),J(t))\gamma'(t)=0$ 這裡有關於測地線偏離的一點敘述。
4. 通過 geodesic 建構 exponential map 建立 normal coordinates 簡化計算。
5. Geodesic flows

   Geodesic equation：$\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^k=0$

Observe that any curve γ determines a curve $t \to (\gamma(t), \gamma'(t))$ in TM. If is a geodesic then in local coordinate $t \to (x_1(t)\ldots, x_n(t), \frac{dx_1(t)}{dt}, \ldots \frac{dx_n(t)}{dt})$ satisfies the first order system

$$\begin{cases} \frac{dx_k}{dt} = y_k \\ \frac{dy_k}{dt} = -\sum_{i,j}\Gamma^k_{ij}y_iy_j \end{cases} \tag{4}$$

on TU.

Lemma 3.3. *There exist a unique vector field G on TM whose trajectories are of the form $t \to (\gamma(t), \gamma'(t))$, where γ is a geodesic on M.*

Definition 3.4. *The vector field defined above is called the geodesic field on TM and its corresponding flow is called the geodesic flow on TM.*

6. geodesic(Wormhole)

§ 09 古典的作法如下：

$C: X = X(s)$是曲面 M 上的一條曲線

$t = \frac{dX}{ds}$是單位切向量，$\frac{dX}{ds} = X_i \frac{du^i}{ds}$則

$\frac{dt}{ds} = \frac{d}{ds}(X_i \frac{du^i}{ds}) = X_i \frac{d^2u^i}{ds^2} + (X_{ij}\frac{du^j}{ds})\frac{du^i}{ds}$，$X_{ij} = \Gamma_{ij}^k X_k + b_{ij}N = \Gamma_{kj}^i X_i + b_{kj}N$

$= (\frac{d^2u^i}{ds^2} + \Gamma_{jk}^i \frac{du^j}{ds}\frac{du^k}{ds})X_i + b_{jk}N$，注意到上式中 $i \leftrightarrow k$，且 $\Gamma_{ij}^k = \Gamma_{ji}^k$

取切部，所以$\frac{d^2u^i}{ds^2} + \Gamma_{jk}^i \frac{du^j}{ds}\frac{du^k}{ds} = 0$---(1)是測地線方程式。

§ 10

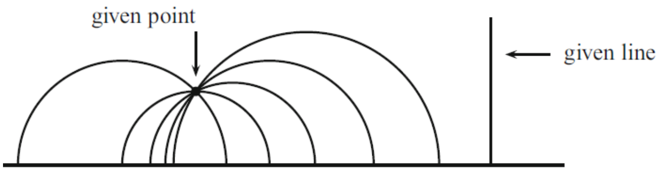


Fig. 14.4. Geodesics in the Poincaré half-plane.

杜武亮 p.108~110

有 Poincare Half-Plane 上測地線的推導。

11.1. Connection and curvature forms on the Poincaré disk

The ***Poincaré disk*** is the open unit disk

$$\mathbb{D} = \{z = x + iy \in \mathbb{C} \mid |z| < 1\}$$

in the complex plane with Riemannian metric

$$\langle\ ,\ \rangle_z = \frac{4(dx \otimes dx + dy \otimes dy)}{(1-|z|^2)^2} = \frac{4(dx \otimes dx + dy \otimes dy)}{(1-x^2-y^2)^2}.$$

An orthonormal frame for $\mathbb{D}$ is

$$e_1 = \frac{1}{2}(1-|z|^2)\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{2}(1-|z|^2)\frac{\partial}{\partial y}.$$

Find the connection matrix $\omega = [\omega_j^i]$ and the curvature matrix $\Omega = [\Omega_j^i]$ relative to the orthonormal frame e_1, e_2 of the Riemannian connection ∇ on the Poincaré disk. (*Hint*: First find the dual frame θ^1, θ^2. Then solve for ω_j^i in (11.9).)

Let D be the Poincare disk

1. Show that in polar coordinates (r,θ)，the Poincare metric is given by

$$\langle .,. \rangle_{(r,\theta)} = \frac{4(dr^2 + r^2 d\theta^2)}{(1-r^2)^2}$$

Using polar coordinates to compute for D

2. The Gaussian curvature
3. The Christoffel symbols
4. The geodesic equations

p.103~ 有 geodesic 古典性質

例如 *The speed of a geodesic on a Riemannian manifold is constant*。

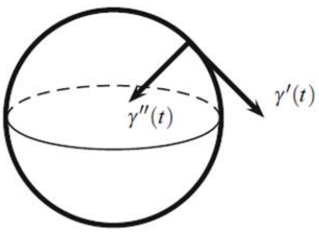
Let $T = c'(t)$ be the velocity of the geodesic。

The speed is constant if and only if its square f (t)=<T , T> is constant。

But $f'(t) = \frac{d}{dt} < T, T >= 2 < \frac{DT}{dt}, T >= 0$

So f(t) is constant。

§ 11 Geodesics on a sphere



Velocity and acceleration vectors of a great circle.

On a 2-sphere M of radius a in R^3

Let $\gamma(t)$ be a great circle parameterized by arc length。

Then $\gamma(t)$ has unit speed。

Differentiating $\langle \gamma'(t), \gamma'(t) \rangle = 1$ with respect to t gives $2\langle \gamma''(t), \gamma'(t) \rangle = 0$

This shows that the acceleration $\gamma''(t)$ of a unit-speed curve in R^3 is perpendicular to the velocity。

Since $\gamma(t)$ lies in the plane of the circle，so do $\gamma'(t)$ and $\gamma''(t)$。

Being perpendicular to $\gamma'(t)$，the acceleration $\gamma''(t)$ must in the radial direction。Hence，because $\gamma''(t)$ is perpendicular to the tangent plane at $\gamma(t)$。

$$\frac{DT}{dt} = \left(\frac{dT}{dt}\right)_{\tan} = \gamma''(t)_{\tan} = 0$$

This shows that every great circle is a geodesic on the sphere。

[Differential Geometry in Physics] p.154 by Gabriel Lugo

Let S^2 be a sphere of radius a，so that the metric is given by

$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2$ and the Lagrangian is $L = a^2 \dot{\theta}^2 + a^2 \sin^2\theta \dot{\varphi}^2$

The E-L equation for the φ coordinate is $\frac{d}{ds}(\frac{\partial L}{\partial \dot{\varphi}}) - \frac{\partial L}{\partial \varphi} = 0$

$\frac{d}{ds}(2a^2 \sin^2\theta \dot{\varphi}) = 0$ then $\sin^2\theta \frac{d\varphi}{ds} = k$

$$\sin^2\theta d\varphi = kds$$

$$\sin^4\theta d\varphi^2 = k^2 ds^2 = k^2(a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2)$$

$$(\sin^4\theta - a^2k^2\sin^2\theta)d\varphi^2 = a^2k^2 d\theta^2$$

$$d\varphi = \frac{akd\theta}{\sin\theta\sqrt{\sin^2\theta - a^2k^2}} = \frac{akd\theta}{\sin^2\theta\sqrt{1 - a^2k^2\csc^2\theta}}$$

$$= \frac{akcsc^2\theta d\theta}{\sqrt{1 - a^2k^2(1+\cot^2\theta)}} = \frac{ak\csc^2\theta d\theta}{\sqrt{(1-a^2k^2) - a^2k^2\cot^2\theta}}$$

$= \frac{\csc^2\theta d\theta}{\sqrt{\frac{1-a^2k^2}{a^2k^2} - \cot^2\theta}}$，let $u = \cot\theta$，$c^2 = \frac{1-a^2k^2}{a^2k^2}$ then $du = -\csc^2\theta d\theta$

$= \frac{-du}{\sqrt{c^2 - u^2}}$ $\left(\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}\right)$

$$\varphi = -\sin^{-1}(\frac{1}{c}\cot\theta) + \varphi_0$$

$$\cot\theta = c\sin(\varphi_0 - \varphi)$$

$$\cos\theta = c\sin\theta(\sin\varphi_0\cos\varphi - \cos\varphi_0\sin\varphi)$$

$$a\cos\theta = (c\sin\varphi_0)(a\sin\theta\cos\varphi) - (c\cos\varphi_0)(a\sin\theta\cos\varphi)$$

$z = Ax - By$ ， where $A = c\sin\varphi_0, B = c\cos\varphi_0$

We conclude that the geodesics of the sphere are great circles determined by the intersections with planes through the origin。

後面有旋轉面的 geodesics

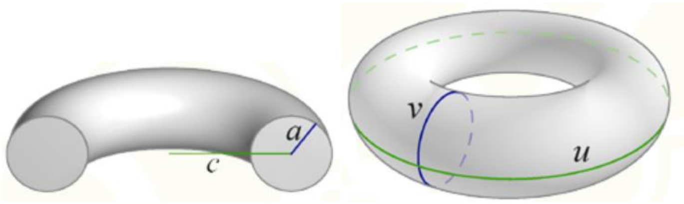
z=f(r) the first fundamental form is $ds^2 = (1+f'^2)dr^2 + r^2 d\varphi^2$

…

然後有一個有趣的蟲洞 Morris-Thorne Wormhole p.159

geodesic(Wormhole)

§ 12 geodesics on a torus



[the torus]

$$\begin{cases} x = c + a\cos v\cos u \\ y = c + a\cos v\sin u \\ z = a\sin v \end{cases}$$

By geodesic equations $\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$ we get

$$\ddot{u} - \frac{2a\sin v}{c + a\cos v}uv = 0$$

$$\ddot{v} + \frac{1}{a}\sin vc + a\cos v\dot{u}^2 = 0$$

Using the substituion w=c+acosv and integrating (with a trick or two)gives a solution in terms of u and v

$$\begin{cases} \dot{u} = \dfrac{k}{c + a\cos v^2} \\ \dot{v} = \pm\sqrt{-\dfrac{k^2}{a^2 c + a\cos v^2} + l} \end{cases}$$

The Clairaut parameterization of a torus treats it as a surface of revolution。

From it，we obtain a formula for $\dfrac{du}{dv}$

$$\frac{du}{dv}=\pm\frac{ah}{c+a\cos v\sqrt{c+a\cos v^{2}-h^{2}}}$$

$$h=c+a\cos v\sin\varphi$$

The five families of geodesics：

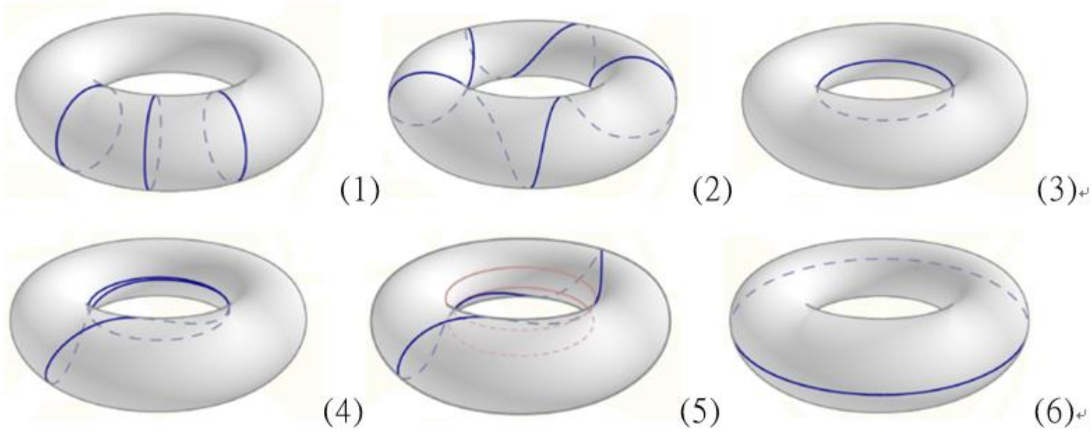
(1) h=0，these are the meridians

(2) 0<h<c-a unbounded geodesics，which alternately cross inner and outer equators。

(3) h=c-a the inner equator ，and geodesics asymptotic to it

(4) c-a<h<c+a bounded geodesics, which cross the outer equator but bounce off barrier curves

(5) h=c-a the outer equator


(1) (2) (3) (4) (5) (6)

When h>c+a，there are no real solutions to the formula for $\frac{du}{dv}$

§ 13 The geodesic equations for a metric of a BH

We consider a charged BH surrounded quintessence with the equation of state parameter $\varepsilon=\frac{p\phi}{\sigma\phi}$，$-1<\varepsilon<-\frac{1}{3}$，and α is the normalization constant。

Quintessence 第五元素 a hypothetical form of dark energy ，a scalar field

The metric of a charged BH reads：

$$ds^{2}=f(r)dt^{2}-\frac{1}{f(r)}dr^{2}-r^{2}(d\theta^{2}+\sin^{2}\theta d\phi^{2})$$

Where $f(r)=1-\frac{2M}{r}+\frac{Q^{2}}{r}-\frac{\alpha}{r^{3\varepsilon+1}}$

Where M and Q represent the mass and charge of the BH respectively。

The metric represents a BH for M>Q，an extremal BH for M=Q and a naked singularity for M<Q。

The geodesic equations for the metric are given by

$$\ddot{t}+\frac{f'(r)}{f(r)}\dot{r}\dot{t}=0 \qquad (1)$$

$$\ddot{r}+[\frac{f'(r)\dot{t}^2+f^{-1}(r)'\dot{r}^2-2r\dot{\theta}^2-2r\sin^2\theta\dot{\phi}^2}{2f^{-1}(r)}]=0 \qquad (2)$$

$$\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\cos\theta\sin\theta\dot{\phi}^2=0 \qquad (3)$$

$$\ddot{\phi}+\frac{2}{r}\dot{r}\dot{\phi}+2\cot\theta\dot{\theta}\dot{\phi}=0 \qquad (4)$$

Where the prime denotes the differentiation with respect to r 。

The geodesic equations are $\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^k=0$

Where $\Gamma^i_{jk}=\frac{1}{2}g^{il}(\frac{\partial g_{kl}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l})$

$$(g_{ij})=\begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & -\frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix},\ (g^{ij})=\begin{pmatrix} \frac{1}{f(r)} & 0 & 0 & 0 \\ 0 & -f(r) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{r^2\sin^2\theta} \end{pmatrix}$$

$$X=X(t,r,\theta,\phi)$$

$$\ddot{t}+\Gamma^1_{jk}\dot{x}^j\dot{x}^k=0$$

$$\ddot{t}+\Gamma^1_{11}\dot{t}\dot{t}+\Gamma^1_{12}\dot{t}\dot{r}+\Gamma^1_{21}\dot{r}\dot{t}+\Gamma^1_{22}\dot{r}\dot{r}+\{...\}$$

$$\Gamma^1_{11}=\frac{1}{2}g^{1l}(\frac{\partial g_{1l}}{\partial x^1}+\frac{\partial g_{1l}}{\partial x^1}-\frac{\partial g_{11}}{\partial x^l})=0\ ,$$

$$\Gamma^1_{22}=\frac{1}{2}g^{1l}(\frac{\partial g_{2l}}{\partial x^2}+\frac{\partial g_{2l}}{\partial x^2}-\frac{\partial g_{22}}{\partial x^l})=\frac{1}{2}g^{11}(\frac{\partial g_{21}}{\partial x^2}+\frac{\partial g_{21}}{\partial x^2}-\frac{\partial g_{22}}{\partial x^1})=0$$

$$\Gamma^1_{12}=\frac{1}{2}g^{1l}(\frac{\partial g_{2l}}{\partial x^1}+\frac{\partial g_{1l}}{\partial x^2}-\frac{\partial g_{12}}{\partial x^l})=\frac{1}{2}\times\frac{1}{f(r)}\times\frac{\partial f}{\partial r}=\Gamma^1_{21}$$

所以得到(1) $\ddot{t}+\frac{f'(r)}{f(r)}\dot{t}\dot{r}=0$

$$\ddot{r}+\Gamma^2_{jk}\dot{x}^j\dot{x}^k=0$$

$$\Gamma^2_{11}=\frac{1}{2}g^{2l}(\frac{\partial g_{1l}}{\partial x^1}+\frac{\partial g_{1l}}{\partial x^1}-\frac{\partial g_{11}}{\partial x^l})=\frac{1}{2}g^{22}\times(-\frac{\partial g_{11}}{\partial x^2})=\frac{1}{2}f(r)\frac{\partial f(r)}{\partial r}$$

$$\Gamma^2_{22}=\frac{1}{2}g^{22}(\frac{\partial g_{2l}}{\partial x^2}+\frac{\partial g_{2l}}{\partial x^2}-\frac{\partial g_{22}}{\partial x^l})=\frac{1}{2}g^{22}(\frac{\partial g_{22}}{\partial r}+\frac{\partial g_{22}}{\partial r}-\frac{\partial g_{22}}{\partial r})=\frac{1}{2}f(r)\times f^{-1}(r)'$$

$$\Gamma^2_{33}=\frac{1}{2}g^{2l}(\frac{\partial g_{3l}}{\partial x^3}+\frac{\partial g_{3l}}{\partial x^3}-\frac{\partial g_{33}}{\partial x^l})=\frac{1}{2}(-2rf(r))$$

$$\Gamma^2_{44}=\frac{1}{2}g^{2l}(\frac{\partial g_{4l}}{\partial x^4}+\frac{\partial g_{4l}}{\partial x^4}-\frac{\partial g_{44}}{\partial x^l})=\frac{1}{2}(-f(r))(2r\sin^2\theta)$$

得到(2)

$$\ddot{\theta}+\Gamma^3_{jk}\,d\dot{x}^j\,d\,\dot{x}^k=0$$

$\Gamma^3_{11}=0$　$\Gamma^3_{22}=0$　$\Gamma^3_{33}=0$　$\Gamma^3_{44}=-\sin\theta\cos\theta$

$\Gamma^3_{23}=\Gamma^3_{32}=\frac{1}{r}$，其他皆為 0

得到(3) $\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\cos\theta\sin\theta\,\dot{\phi}^2=0$

$$\ddot{\phi}+\Gamma^4_{jk}d\,\dot{x}^j\,d\,\dot{x}^k=0$$

$\Gamma^4_{24}=\Gamma^4_{42}=\frac{1}{r}$　　$\Gamma^4_{34}=\Gamma^4_{43}=\cot\theta$，其他皆為 0

得到(4) $\ddot{\phi}+\frac{2}{r}\dot{r}\dot{\phi}+2\cot\theta\,\dot{\theta}\dot{\phi}=0$

§ 14 參考書目

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