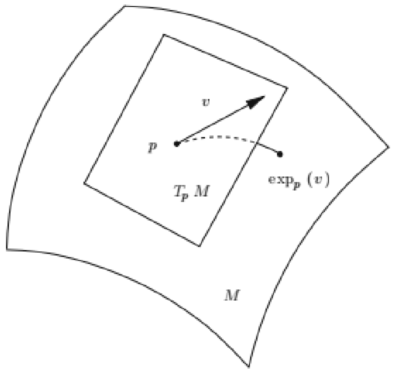
§ Exponential Map

There are two kinds of exponential map：

The exponential map of a connection and a Lie group。

When a Lie group has a bi-invariant Riemannian metric，as all compact Lie groups do，the exponential map for the Lie group coincides with the exponential map of the Riemannian connection。

§ The exponential map of a connection，denoted as Exp_p



The exponential map

M is a manifold with an affine connection

$\gamma_V(t)$是從 $p=\gamma_V(0)$出發、切於 V 的測地線 (即$\gamma_V(0)=p, \frac{d\gamma_V}{dt}(0)=V$)

給定 $p\in M$ ，定義$Exp_p:T_pM\to M$ 使得 $Exp_pV=\gamma_V(1), \forall V\in T_pM$

(可以導出$\gamma_X(t)=Exp_p(tX), \forall t$)

That is defined locally。If the exponential map well-defined at every point of the tangent bundle，then the affine connection called complete。

The range can fail to be all of M simply because there can be two points that are not connected by any geodesic。

The domain can fail to be all of T_p because a geodesic may run into a singularity。

Manifolds that have such singularities are known as geodesically incomplete。

[An Introduction to Riemannian Geometry p.110]有 normal neighborhood 的命題，然後定義 normal ball　p.115 習作 3.4.8 有 normal coordinates 的定義。

§ normal neighborhood

$Exp_p : T_pM \to M$ is a local diffeomorphism

$X \to \gamma(1)$ where X is the tangent vector of the geodesic γ at p。

$X = \frac{d\gamma}{dt}(0), Exp_p(0) = \gamma(0) = p$

Then ∃ V(open) in T_pM ，such that $Exp_p : V \to U$ is a diffeomorphism，U is called a normal neighborhood of p。

§ Normal coodinates

Fix a point p in a Riemannian manifold M。

There is a neighborhood V of 0 in T_pM and a neighborhood U of p in M such that the exponetial map $Exp_p : V \to U$ is a diffeomorphism。

Using the exponential map we can tramsfer coordinates on T_pM to M。

Choose an orthonormal basis $e_1,...,e_n$ for T_pM and let $r^1,...,r^n$ be the coordinates with respect to the orthonormal basis $e_1,...,e_n$ on T_pM 。

Then $x^1 := r^1 \circ Exp_p^{-1},...,x^n := r^n \circ Exp_p^{-1}$ is a coordinates system on U such that the tangent vectors $\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n}$ are orthonormal at p。

The coordinate neighborhood $(U, x^1,...,x^n)$ is called a normal neighborhood of p and $x^1,...,x^n$ are called normal coordinates on U。

In a normal neighborhood of p，the geodesics througth p have a particularly simple expression，for the coordinate expression for the geodesic $\gamma(t) = Exp_p(at)$ for $a = \sum a^i e_i \in T_pM$ is $x(\gamma(t)) = r \circ Exp_p^{-1}(\gamma(t)=at$

We write this as $(x^1,...,x^n) = (a^1t,...,a^nt)$

Exercse 3.4.8

(2) Let M be a Riemannian manifold and ∇ the Levi–Civita connection on M. Given $p \in M$ and a basis $\{v_1, \ldots, v_n\}$ for T_pM, we consider the parameterization $\varphi : U \subset \mathbb{R}^n \to M$ of a normal neighborhood given by

$$\varphi(x^1, \ldots, x^n) = \exp_p(x^1 v_1 + \cdots + x^n v_n)$$

(the local coordinates $(x^1, \ldots, x^n)$ are called **normal coordinates**).
Show that:

(a) in these coordinates, $\Gamma^i_{jk}(p) = 0$ (**Hint:** Consider the geodesic equation);
(b) if $\{v_1, \ldots, v_n\}$ is an orthonormal basis then $g_{ij}(p) = \delta_{ij}$.

指數映射用來定義常態座標(Riemannian normal coordinates，or locally inertial coordinates)以簡化計算。

§ Theorem

In a normal neighborhood $(U, x^1, ..., x^n)$ of p，all the partial derivatives of g_{ij} and all the Christoffel symbols Γ^i_{jk} vanish at p。

Let $(x^1, ..., x^n) = (a^1 t, ..., a^n t)$ be a geodesic through p。

It satisfies the geodesic equations $\ddot{x}^i + \sum_{j,k} \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$ or $\sum \Gamma^i_{jk} a^j a^k = 0$

Since this is true for all $(a^1, ... a^n)$ at p。Setting $(a^1, ... a^n) = (0, ...1, 0, ...0, 1, ...0)$

with $a^j = a^k = 1$ and all other entries 0，we get $\Gamma^i_{jk} + \Gamma^i_{kj} = 0$

by the symmetry of the connection，$\Gamma^i_{jk} = 0$ at p。(At other points，not all of $(x^1, ..., x^n) = (a^1 t, ..., a^n t)$ will be geodesics。)

by the compatibiity of the connection ∇ with the metric，

$$\partial_k g_{ij} = \partial_k < \partial_i, \partial_j > = < \nabla_{\partial_k} \partial_i, \partial_j > + < \partial_i, \nabla_{\partial_k} \partial_j >$$

At p，$\nabla_{\partial_k}\partial_i = 0$ and $\nabla_{\partial_k}\partial_j = 0$ since all the Christofell symbols vanish。

Therefore，$(\partial_k g_{ij})(p) = 0$

Normal coordinates are especially useful for computution，because at the point p，all $\nabla_{\partial_k}\partial_i = 0$

在一個黎曼流形(M,g) 因為存在 totally normal neighborhoods，因此可以用極小化的測地線連接 M 上距離夠小的兩點，因此而有 Hopf-Rinow 定理。

Gauss lemma

在 normal neighborhood 內 其邊界$\partial B_p(r)$與從 p 點出發的測地線垂直。

證明在 p.111~112

§ Left-invariant vector field on a Lie group

$\eta = T_e G$ is the Lie algebra of Lie group G

$$\begin{array}{ccc} TM & \xrightarrow{df} & TN \\ X\uparrow & & \uparrow f_* X \\ M & \xrightarrow{f} & N \end{array}$$

$(f_* X)_{f(p)} := (df)_p X_p$ ，the map is a diffeomorphism

$$\begin{aligned} L_g : G &\to G \\ h &\mapsto g\cdot h \end{aligned}$$ Is called left-muptiplication

For a induced map $(L_g)_* : T_h(G) \to T_{gh}(G)$ ，a vector field X is called left-invariant

$\Leftrightarrow (L_g)_* X = X$ for all $g \in G$ ，that is $((L_g)_* X)_{gh} = X_{gh}$

or $(dL_g)_h X_h = X_{gh}$ for all $g, h \in G$

let { left-invariant vector fields on G }= $\chi_L(G)$ ，then $\chi_L(G) \cong \eta$

for each $V \in \eta$ ， X_g^V:=$(\mathrm{dL}_g)_e V$ for any $g \in G$

Then the vector field X_h^V is left-invariant

$$(dL_g)_h X_h^V = (dL_g)_h (dL_h)_e V = (d(L_g \circ L_h))_e V = (dL_{gh})_e V = X_{gh}^V$$

§ The local flow φ_t of a left-invariant vector field X on a Lie group G commutes with left-multiplication $L_g \circ \varphi_t = \varphi_t \circ L_g$ for all $g \in \mathrm{G}$ ，whenever both sides are defined。

§ The exponential map of a Lie group，denoted as exp

Let G is a Lie group， $\mathrm{X} \in \eta$ 。Let X_γ be the integral curve of X starting at the identity。Then the exponential map $\eta \to \mathrm{G}$ ， $\exp X = \gamma_X(1)$

The exponential map $\exp : \eta \to G$ is the map that，to each $V \in \eta$ assigns the value $\psi_1(e)$ ，where ψ_t is the flow of the left-invariant vector field X^{V}

If G is a group of matrices，then for $\mathrm{A} \in \eta$ $\exp A = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

$h(t) = e^{At}$ satisfies (1) $h(0) = e^0 = I$ (2) $\frac{dh}{dt} = e^{At} A = h(t) A$

h(t) is the flow of X^A at the identity and so $\exp A = \psi_1(e) = e^A$

Proposition 15.9. (i) *For $X_e \in \mathfrak{g}$, the integral curve starting at e of the left-invariant vector field X is* $\exp(tX_e) = c_X(t) = \varphi_t(e)$.

(ii) *For $X_e \in \mathfrak{g}$ and $g \in G$, the integral curve starting at g of the left-invariant vector field X is* $g\exp(tX_e)$.

(iii) *For $s,t \in \mathbb{R}$ and $X_e \in \mathfrak{g}$,* $\exp\left((s+t)X_e\right) = (\exp sX_e)(\exp tX_e)$.

(iv) *The exponential map* $\exp\colon \mathfrak{g} \to G$ *is a* C^∞ *map.*

(v) *The differential at 0 of the exponential map,* $\exp_{*,0}\colon T_0(\mathfrak{g}) = \mathfrak{g} \to T_eG = \mathfrak{g}$ *is the identity map.*

(vi) *For the general linear group* $\mathrm{GL}(n,\mathbb{R})$,

$$\exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad \textit{for } A \in \mathfrak{gl}(n,\mathbb{R}).$$

(proof followed) 杜武亮 p.120

Let $\phi: R \to G$ be a smooth Lie group homomorphism，and let $A = \phi'(0) \in T_eG$ be a tangent vector at the identity。

(R 是加法群，G 是乘法群 $\therefore \phi(0) = I$)

Such homomorphism is called an one-parameter subgroup of G。

Let G=GL(n, R) this a matrix group，so the tangent vector at identity is a matrix。

Since $\phi(s+t) = \phi(s)\phi(t)$，evaluating the derivative at s=0 gives

$$\left(\frac{d\phi}{dt} = \lim_{h\to 0}\frac{\phi(t+h)-\phi(t)}{h} = \lim_{h\to 0}\frac{\phi(t)\phi(h)-\phi(t)}{h} = \phi(t)\lim_{h\to 0}\frac{\phi(h)-I}{h} = \phi'(0)\phi(t)\right)$$

$\frac{d}{dt}\phi(t) = \phi'(0)\cdot\phi(t) = A\phi(t)$，with $\phi(0) = I$

Then $\phi(t) = e^{At}$

So for any Lie group G，we define the exponential map $\exp: \eta \to G$

$\exp(X) = \gamma_X(1)$ is a diffeomorphism

Let $A \in T_eG$，and $\phi: R \to G$ be the unique homomorphism such that $\phi'(0) = A$

Then $\exp(A) = e^A = \phi(1)$

The map $t \to e^{tA}$ is a local diffeomorphism from T_eG to G。

例 G=SO(3, R) then $T_e G = so(3,R)$

$$J_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, J_Y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, J_Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\exp : so(3,R) \to SO(3,R)$

$$\exp(tJ_X) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$ rotation around x-axis by angle t

$$\exp(tJ_Y) = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix}$$ rotation around y-axis by angle t

$$\exp(tJ_Z) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ rotation around z-axis by angle t

參考書目

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