§ 流形上有三個微分運算

(1) 外微分 d：針對 differential forms

(2) Lie derivative $L_X Y$ or $L_X \omega$

(3) covariant derivative

$X=\sum_i X^i \frac{\partial}{\partial x^i}, Y=\sum_i Y^i \frac{\partial}{\partial x^i}$ ，φ_t is the flow a vector field X，Y is a C^∞ vector field，then the Lie derivative of Y along X is $L_X Y=\lim_{t\to 0}\frac{(\varphi_{-t})_* Y - Y}{t}=\frac{d}{dt}((\varphi_{-t})Y)\big|_{t=0}$ =[X,Y]

Lie derivative of a form ω

$X\in\chi(M)$ ，$L_X\omega := \lim_{t\to 0}\frac{1}{t}(\varphi_t^*\omega-\omega)=\frac{d}{dt}(\varphi_t^*\omega)\big|_{t=0}$ ，Where φ_t is the local flow of X

Covariant derivative $\nabla_X Y$ ：Y 在 X 方向的導數取切部。

其中 covariant derivtive for vector fields

$X=\sum_i X^i \frac{\partial}{\partial x^i}, Y=\sum_i Y^i \frac{\partial}{\partial x^i}$ ，則$\nabla_X Y=\sum_i (XY^i+\sum_{j,k}\Gamma^i_{jk}X^jY^k)\frac{\partial}{\partial x^i}$

寫成$\nabla_\mu V^\nu=\partial_\mu V^\nu+\Gamma^\nu_{\mu\sigma}V^\sigma$

Covariant derivative of a 1-form ω

$\nabla_X\omega=\sum_i (X\omega_i-\sum_{j,k}\Gamma^k_{ji}X^j\omega_k)dx^i$ ，寫成$\nabla_\mu\omega_\nu=\partial_\mu\omega_\nu-\Gamma^\lambda_{\mu\nu}\omega_\lambda$

例如 metric compatibility$\Leftrightarrow\nabla_\rho g_{\mu\nu}=0$

$L_X\omega=\lim_{t\to 0}\frac{1}{t}(\varphi_t^*\omega-\omega)=\frac{d}{dt}\left(\varphi_t^*\right)\big|_{t=0}$ ，then

1. $L_X(\omega\wedge\eta)=(L_X\omega)\wedge\eta+\omega\wedge(L_X\eta)$

2. $d(L_X\omega)=L_X d\omega$

3. $L_X\omega=\iota_X d\omega+d(\iota_X\omega)$

4. $L_X(\iota_Y\omega)=\iota_{L_X Y}\omega+\iota_Y L_X\omega$

5. Let $\omega=\sum_i h_i dx^i, X=\sum_i \xi^i \frac{\partial}{\partial x^i}$ ，then $L_X\omega=\sum_j (Xh_j)dx^j+\sum_k h_k d\xi^k$

Prove

1. $L_X(\omega\wedge\eta)=\frac{d}{dt}\varphi_t^*(\omega\wedge\eta)\Big|_{t=0}=\frac{d}{dt}(\varphi_t^*\omega)\wedge(\varphi_t^*)\Big|_{t=0}$

$=(\frac{d}{dt}(\varphi_t^*\omega)\big|_{t=0})\wedge\eta+\omega\wedge(\frac{d}{t}(\varphi_t^*\eta)\big|_{t=0})=(L_X\omega)\wedge\eta+\omega\wedge(L_X\eta)$

2. …
3. …
4. …
5. (Let $\omega=\omega_i d\omega^i, X=X^j\frac{\partial}{\partial x^j}$ ，then $L_X\omega=(X^j\frac{\partial\omega_k}{\partial x^j}+\omega_j\frac{\partial x^j}{\partial x^k})dx^k$)

§ Cartan magic formula $L_X\omega=\iota_X d\omega+d(\iota_X\omega)$

右式前者是 interior product(interior derivatve)，後者是 exterior derivative
外微分後做內積 + 做內積後再外微分。

1. 例子

$X=F\frac{\partial}{\partial x}+G\frac{\partial}{\partial y}+H\frac{\partial}{\partial z}$ ，volume form $\omega=dv=dx\wedge dy\wedge dz$

$L_X dx=d(L_X x)=d(Xx)=dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz$

Then
左式

$L_X dv=L_X(dx\wedge dy\wedge dz)=(L_X dx)\wedge dy\wedge dz+dx\wedge(L_X dy)\wedge dz+dx\wedge dy\wedge(L_X dz)$

$=(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z})dv=(divX)dv$

右式

$d\omega=d(dv)=0$

$\iota_X\omega=Fdy\wedge dz-Gdx\wedge dz+Hdx\wedge dy$

$d(\iota_X\omega)=dF\wedge dy\wedge dz-dG\wedge dx\wedge dz+dH\wedge dx\wedge dy$

其中 $dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz$ ，…。

所以 $d(\iota_X \omega) = (\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}) dx \wedge dy \wedge dz = (div X) dv$

所謂的 interior product 如何運算：

$$\iota_X (dx_1 \wedge dx_2 \ldots \wedge dx_n) = \sum_{r=1}^{n} (-1)^{r-1} X^r dx_1 \wedge \ldots \wedge \hat{dx_r} \wedge \ldots \wedge dx_n$$

這裡 $\hat{dx_r}$ 表示把 dx_r 省略。

又 $\eta = F dy \wedge dz + G dz \wedge dx + H dx \wedge dy$ 直接計算

$d\eta = (div X) dx \wedge dy \wedge dz = L_X dv$ 這是顯然的，因為 $\eta = \iota_X \omega$ 。

2. 證明

[Elementary Proof of the Cartan Formula]

3. 應用

X is a Killing vector field $\Leftrightarrow L_X g = 0$

$$(L_X g)(Z, W) = g(\nabla_Z X, W) + g(Z, \nabla_W X)$$

X is a harmonic vector field $\Leftrightarrow dX^b = 0$ and div X=0