§ Killing vector field

Wilhelm Killing 1847-1923

(M,g) is a Riemann manifold。

Killing vector field 是定義在離曼流形上的向量場，流形的度規在向量場的方向上能夠保持不變。

A vector field X is a Killing field $L_X g = 0$。

(大概就是說，g 在 X 方向的方向導數=0，所以 g 在這個方向為常數，只是此處是 Lie derivative。)

另一種微分是 covariant derivatuve，用它的話：

In terms of the Levi-Civita connection，that is

$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all vectors Y and Z。

Or $\nabla_\mu K^\nu = -\nabla_\nu K^\mu$ in local coordinates。

A vector field $K = K^\mu \partial_\mu$ on M is said to be a Killing vector field if the infinitesimal

displacement $\varphi : x^\mu \to x^\mu + \varepsilon K^\mu$ generates an isometry。

Show that this is the case，if $X^\kappa \partial_\kappa g_{\mu\nu} + \partial_\mu X^\kappa g_{\kappa\nu} + \partial_\nu X^\kappa g_{\mu\kappa} = 0$

These are the so-called Killing equation。

Show that the Killing equations can be written as $L_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$

研究一個度規(metric)的等距同構(isometries)的主要工具是 Killing 向量場。

In GR，the word symmetry is associated to that of isometry，that is a spacetime diffeomorphism that leaves the metric invariant。

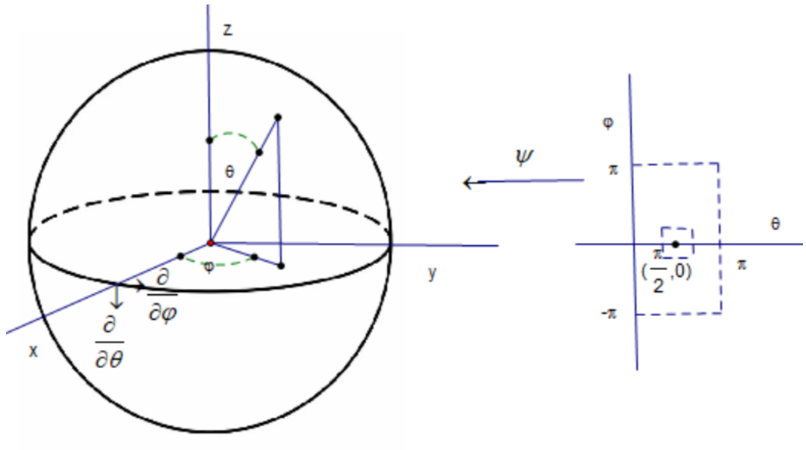
A one-parameter continuous isometry is linked to the existence of Killing vectors。

Much of the activity in the area related to using symmetries to solve Einstein equation or the equations of motion of other systems has been directed towards finding metrics admitting Killing vectors。

Hidden Symmetries of Dynamics in Classical and Quantum Physics by Marco Cariglia

Killing tensors are not related in a simple way to symmetries of the spacetime，but they will simplify our analysis of rotating black holes and expanding universes。

§ S^2 上的 Killing vector field



$\psi:(0,\pi)\times(-\pi,\pi)\to S^2$ given by $\psi(\theta,\varphi)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$

Parameterizes a neighorhood of the point $(1,0,0)=\psi(\frac{\pi}{2},0)$

我們注意到$\psi(\theta,0)\to$通過(1,0,0)的經線，$\psi(0,\varphi)\to$赤道。

$$ds^2=d\theta^2+\sin^2\theta d\varphi^2$$

$$x=\sin\theta\cos\varphi, y=\sin\theta\sin\varphi, z=\cos\theta$$

$$\frac{\partial}{\partial\varphi}=\frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x}+\frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y}+\frac{\partial z}{\partial\varphi}\frac{\partial}{\partial z}$$

$$=-\sin\theta\sin\varphi\frac{\partial}{\partial x}+\sin\theta\cos\varphi\frac{\partial}{\partial y}+0=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$$

$\frac{\partial}{\partial\varphi}=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$ is a vector field which generates rotation about the z-axis，is an isometry and a Killing vector field that preserves the metric，i.e. $L_X g=0$。

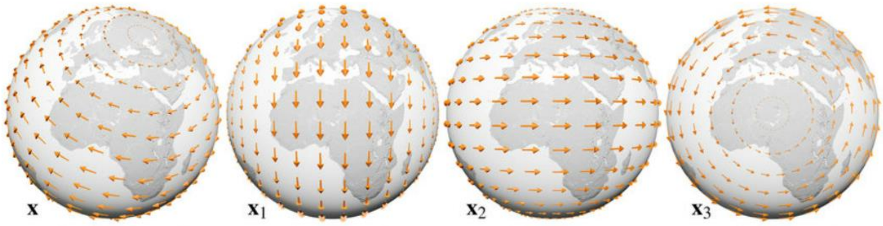
The flow of $\frac{\partial}{\partial\varphi}=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$ is $\varphi_t=(x\cos t-y\sin t, x\sin t+y\cos t, z)$，and the matrix of the rotation about the z-axis is $\begin{pmatrix}\cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{pmatrix}$

$$\{\varphi_t | t\in R\}\cong SO(2,R)$$

$$\frac{d}{d\theta}\Big|_{\theta=0}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-y\partial_x+x\partial_y$$

But $\frac{\partial}{\partial\theta}=\cos\theta\cos\varphi\frac{\partial}{\partial x}+\cos\theta\sin\varphi\frac{\partial}{\partial y}-\sin\theta\frac{\partial}{\partial z}$ is not a Killing vector field。

$(\frac{\partial}{\partial\theta})_{(1,0,0)}\leftrightarrow(0,0,-1)$朝下的向量



$X=\begin{pmatrix}0 & -c & b\\ c & 0 & -a\\ -b & a & 0\end{pmatrix}$ $X_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$ $X_2=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}$ $X_3=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

Killing vector fields give the **infinitesimal isometries** of a manifold M，here the sphere S^2 。

X_1, X_2, X_3即是 Lie algebra so(3)中的無窮小生成元(infinitesimal generators)，

$[X_i, X_j]=\varepsilon_{ijk}X_k$ ，ε_{ijk} 稱為 Levi-Civita symbols

$$\frac{d}{d\theta}\Big|_{\theta=0}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=-y\partial_x+x\partial_y$$

We expect S^2 to have **symmetry** under the action of SO(3)。

$Z=-y\partial_x+x\partial_y=\partial\varphi$ ，$X=-z\partial_y+y\partial_z$ ，let $[Z,X]=-z\partial_x+x\partial_z=Y$

Then $[X,Y]=Z,[Y,Z]=X$

Span{X, Y, Z} = Lie algebra so(3)

X、Y、Z are the Killing fields that generate so(3)。

Since $L_{[X,Y]}=L_XL_Y-L_YL_X=0$ ，we can find a third Killing vector by taking the commutator of the first two，given that X and Y are independent Killing vector fields，i.e. $[X,Y]\neq 0$

(2) Let $\mathbb{R}^2$ be equipped with the standard flat metric $dx^2 + dy^2$. Restricting it to the suitable domain $\mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 \mid x < 0\}$, we may rewrite it using polar coordinates (r,θ), and $dx^2 + dy^2 = dr^2 + r^2\, d\theta^2$. So ∂_θ is a Killing field, and the isometries it generates are the rotations around the origin, given by the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

as θ ranges over $[0, 2\pi[$.

The Lie algebra elements X， $X_i \in$ so(3) (bottom) generate the Killing fields x, x_i on M (top) through their Lie algebra action。

Like the matrices X_i，the Killing fields x_i are linearly independent，forming a basis of the Lie algebra of Killing fields on S^2。

That is，as we can expand $X = aX_1 + bX_2 + cX_3$，we likewise get

$x = ax_1 + bx_2 + cx_3$，where the latter means point-wise addition of vectors in each $T \times M$ at each x ∈ M。

Proposition

Let X，Y be two Killing fields，then [X，Y] is also a Killing field。

Thus the space of Killing fields is a Lie algebra，denoted by $iso(M, g)$。In particular，we see that the dimension of iso(M,g) is at most $\dfrac{n(n+1)}{2}$。

Proof

Just directly compute $L_{[X,Y]} \langle\cdot,\cdot\rangle = L_X(L_Y\langle\cdot,\cdot\rangle) - L_Y(L_X\langle\cdot,\cdot\rangle) = L_X(0) - L_Y(0) = 0$

Example

Consider the usual Euclidean space $(R^3, dx^2 + dy^2 + dz^2)$，there are at most 6 linearly independent Killing fields。

- $T_s(x, y, z) = (x + s, y, z)$ is a 1-parameter family of isometries, since we have that $DT_s(x, y, z) = \mathrm{Id}_{\mathbb{R}^3}$. Thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} T_s(x, y, z) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (x + s, y, z) = \partial_x$$

is a Killing field (we already knew that). Similarly we recover that ∂_y and ∂_z are Killing fields.

- We have three 1-parameter families of rotations, around each axis, and the associated Killing fields are given by:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -z\partial_y + y\partial_z,$$

and similarly:

$$\left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -z\partial_x + x\partial_z$$

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Proposition

Proposition. *Let $\boldsymbol{\xi}$ be a Killing field of constant length. Then the integral curves of $\boldsymbol{\xi}$ are geodesics of* $(M, \langle \cdot, \cdot \rangle)$.

Proof: The hypothesis says that $\boldsymbol{X}\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = 0$ for all $\boldsymbol{X} \in \mathfrak{X}(M)$. But this equals $2\langle \nabla_{\boldsymbol{X}}\boldsymbol{\xi}, \boldsymbol{\xi} \rangle = 0$, and it follows that $\nabla_{\boldsymbol{X}}\boldsymbol{\xi}$ is always orthogonal to $\boldsymbol{\xi}$. Now, Killing's equation reads $0 = (\mathscr{L}_{\boldsymbol{\xi}}\langle \cdot, \cdot \rangle)(\boldsymbol{\xi}, \boldsymbol{X}) = \langle \nabla_{\boldsymbol{\xi}}\boldsymbol{\xi}, \boldsymbol{X} \rangle + \langle \boldsymbol{\xi}, \nabla_{\boldsymbol{X}}\boldsymbol{\xi} \rangle = \langle \nabla_{\boldsymbol{\xi}}\boldsymbol{\xi}, \boldsymbol{X} \rangle$. Non-degeneracy of the metric finally says that $\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi} = \mathbf{0}$, as wanted. □

Proposition. *For any Killing field $\boldsymbol{\xi}$, we have* $\operatorname{div} \boldsymbol{\xi} = 0$.

Proof: The right side of coordinate expression $\xi^j{}_{;j} = g^{ij}\xi_{i;j}$ for the divergence of $\boldsymbol{\xi}$ is both symmetric and skew-symmetric in i and j, hence vanishes. □

Remark. Another proof is noting that $\operatorname{div} \boldsymbol{\xi}$ is the trace of the skew-symmetric endomorphism $\nabla\boldsymbol{\xi}$.

With this, we can try to understand one more question: when are Killing fields gradient fields?

Proposition. *Let ξ be a Killing field and assume that $\xi = \operatorname{grad} f$ for some smooth function $f\colon M \to \mathbb{R}$. Then $\nabla\xi = 0$ and $\triangle f = 0$.*

Proof: In the conditions of the statement, we have that $\langle \nabla_X \xi, Y\rangle = \operatorname{Hess}(f)(X,Y)$ for any vector fields $X, Y \in \mathfrak{X}(M)$. This is skew-symmetric in X and Y since $\nabla\xi$ is skew-adjoint. On the other hand, it is also symmetric, since torsion-free connections produce symmetric Hessian tensors. So $\langle \nabla_X \xi, Y\rangle = 0$ for all Y implies that $\nabla_X \xi = \mathbf{0}$ for all X, and so $\nabla\xi = 0$. On the other hand, since $\operatorname{Hess}(f) = 0$, taking the trace we obtain $\triangle f = 0$ as well. ☐

Proposition. *Assume that M is compact, Riemannian and oriented. If ξ is a Killing field and there is a smooth function $f\colon M \to \mathbb{R}$ such that $\xi = \operatorname{grad} f$, then f is constant and $\xi = \mathbf{0}$.*

Proof: We have that $\triangle f = \operatorname{div}\operatorname{grad} f = \operatorname{div}\xi = 0$, so that[3]

$$\triangle(f^2) = 2f\triangle f + 2\|\operatorname{grad} f\|^2 = 2\|\xi\|^2,$$

and thus

$$0 \overset{(*)}{=} \int_M \triangle(f^2)\,\mathrm{d}M = \int_M 2\|\xi\|^2\,\mathrm{d}M \implies \|\xi\| = 0 \implies \xi = \mathbf{0},$$

as wanted, where $(*)$ follows from Stokes' Theorem and the general divergence expression[4] in terms of the volume form $\mathrm{d}M$: $\mathrm{d}(\iota_X \mathrm{d}M) = (\operatorname{div} X)\,\mathrm{d}M$ for any vector field $X \in \mathfrak{X}(M)$. ☐

Proposition (Kostant formula). *If ξ is Killing, then $\nabla_X(\nabla\xi) = R(X,\xi)$, for all fields $X \in \mathfrak{X}(M)$.*

看 Ivo Terek p.7

Corollary. *If ξ is a Killing field and $\gamma\colon [a,b] \to M$ is a geodesic, then ξ restricts to a Jacobi field along γ.*

Proof: Compute:

$$R(\gamma',\xi)\gamma' = (\nabla_{\gamma'}(\nabla\xi))\gamma' = \nabla_{\gamma'}\nabla_{\gamma'}\xi - \nabla_{\nabla_{\gamma'}\gamma'}\xi = \frac{D^2\xi}{\mathrm{d}t^2},$$

using that $\nabla_{\gamma'}\gamma' = \mathbf{0}$.

☐

1. theory of cosmic inflation Alan Guth1947-
2. Notes on Killing fields Ivo Terek
3. Exercises of Killing field
4. The principle tool for imvestigating the isometries of a metric is the Killing vector field。No matter the coordinate system in which the metric is cast，its set of Killing vectors (modulo coordinate transformations) will be the same。

[Redshift and Killing vectors]

5. Why are Killing fields relevant in physics?
6. Noether Symmetry vs Killing vectors and Isometries of Spacetime
7. There are no unit Killing vector fields on even-dim spheres $S^{2n}(\mathrm{C})$。

However，there are odd-dim spheres $\mathrm{S}^{2n+1}(C)$ as well as odd-dim ellipsoids。

Exercises

1. Consider the Killing vector fields on $M = S^2$ with metric $g = d\theta^2 + \sin^2\theta d\phi^2$。

$K_1 = \partial_\phi$，$K_2 = -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi$，$K_3 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi$

Verify that they satisfy the Killing equations。

2. For the metric $ds^2 = A(z)^2(-dt^2 + dx^2 + dy^2) + dz^2$

$T := \frac{\partial}{\partial t}, X := \frac{\partial}{\partial x}, Y := \frac{\partial}{\partial y}$ are Killing vectors。

Hint check that $(L_X g)_{\mu\nu} = X^\rho\partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu X^\rho + g_{\rho\mu}\partial_\nu X^\rho = 0$

3. For any Killing field ξ，$div\xi = 0$

Proof: The right side of coordinate expression $\xi^j{}_{;j} = g^{ij}\xi_{i;j}$ for the divergence of ξ is both symmetric and skew-symmetric in i and j, hence vanishes. □

Remark. Another proof is noting that div ξ is the trace of the skew-symmetric endomorphism $\nabla\xi$.

4. If ξ is Killing then，$\nabla_X(\nabla\xi) = R(X,\xi)$ for all X

5. Show that any Killing vector K^μ satisfies (1) $\nabla_\mu\nabla_\sigma K^\rho = R^\rho_{\sigma\mu\nu}K^\nu$ (2) $K^\lambda\nabla_\lambda R = 0$

Hint (1) derivatives of Killing vectors (2) along with the Bianchi identity and Killing equation and (1)

Proposition. *Let ξ be a Killing field and assume that $\xi = \operatorname{grad} f$ for some smooth function $f\colon M \to \mathbb{R}$. Then $\nabla\xi = 0$ and $\triangle f = 0$.*

Proof: In the conditions of the statement, we have that $\langle \nabla_X \xi, Y\rangle = \operatorname{Hess}(f)(X,Y)$ for any vector fields $X, Y \in \mathfrak{X}(M)$. This is skew-symmetric in X and Y since $\nabla\xi$ is skew-adjoint. On the other hand, it is also symmetric, since torsion-free connections produce symmetric Hessian tensors. So $\langle \nabla_X \xi, Y\rangle = 0$ for all Y implies that $\nabla_X \xi = \mathbf{0}$ for all X, and so $\nabla\xi = 0$. On the other hand, since $\operatorname{Hess}(f) = 0$, taking the trace we obtain $\triangle f = 0$ as well. □

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Proposition. *Assume that M is compact, Riemannian and oriented. If ξ is a Killing field and there is a smooth function $f\colon M \to \mathbb{R}$ such that $\xi = \operatorname{grad} f$, then f is constant and $\xi = \mathbf{0}$.*

Proof: We have that $\triangle f = \operatorname{div}\operatorname{grad} f = \operatorname{div}\xi = 0$, so that[3]

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as wanted, where $(*)$ follows from Stokes' Theorem and the general divergence expression[4] in terms of the volume form $\mathrm{d}M$: $\mathrm{d}(\iota_X \mathrm{d}M) = (\operatorname{div} X)\,\mathrm{d}M$ for any vector field $X \in \mathfrak{X}(M)$. □

7.

8. Let M be a compact Riemannian manifold of even dimension whose sectional curvature is positive。Prove that every Killing field X on M has a singularity (there exists a $p \in M$ such that X(p)=0)

Hint: Let $f\colon M \to R$ be the function $f(q) = \langle X, X\rangle(q)$, $q \in M$, and let $p \in M$ be a minimum point of of f (Cf. the previous Exercise). Suppose that $X(p) \neq 0$. Define a linear mapping $A\colon T_pM \to T_pM$ by $A(y) = A_X Y = \nabla_Y X$, where Y is an extension of $y \in T_pM$. Let $E \subset T_pM$ be orthogonal to $X(p)$. Use the previous exercise to show that $A\colon E \to E$ is an anti-symmetric isomorphism. This implies that $\dim E = \dim M - 1$ is even, which is a contradiction; thus $X(p) = 0$.

附錄

§ Isometry 等距同構

(M,g) is a Rieannian manifold with Levi-Civita connection ∇。

ϕ:M $\to$ M is an isometry $\Leftrightarrow \phi^* g_{\phi(p)} = g_p$

$g_{\phi(p)}(\phi_* X, \phi_* Y) = g_p(X,Y)$ for all $X, Y \in T_pM$

Or in coordinates $\frac{\partial y^{\alpha}}{\partial x^{\mu}}\frac{\partial y^{\beta}}{\partial x^{\nu}}g_{\alpha\beta}(\phi(p))=g_{\mu\nu}(p)$

Where the diffeomorphism ϕ is an isometry (討論 GR 中的等距同構)，preserve the metric。

The flows generated by Killing fields are continuous isometries on the manifold。

Isometries naturally form a group。

A Killing vector satisfies $\nabla_{(\mu K\nu)}=0$ ，and that implies that $K_{\nu}p^{\nu}$ is conserved along a geodesic。 p^{ν} is a 4-momentum。

§

$$X=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}=\frac{\partial}{\partial\varphi}\qquad Y=\frac{\partial}{\partial\theta}=\cos\theta\cos\varphi\frac{\partial}{\partial x}+\cos\theta\sin\varphi\frac{\partial}{\partial y}-\sin\theta\frac{\partial}{\partial z}$$

$$g=dx^2+dy^2+dz^2=d\theta^2+\sin^2\theta d\varphi^2$$

$$(L_X g)_{\mu\nu}=X^{\rho}\partial_{\rho}g_{\mu\nu}+\partial_{\mu}X^{\rho}g_{\rho\nu}+\partial_{\nu}X^{\rho}g_{\rho\mu}$$

經過小心計算結果

$(L_X g)_{\mu\nu}=0,(L_Y g)_{22}=2\sin\theta\cos\theta$ ，所以 X 是 Killing field，Y 不是。

§ Lie derivative of a differential form

$$X=X^j\frac{\partial}{\partial x^j},\omega=\omega_i d\omega^i$$

Then $L_X\omega=(X^j\frac{\partial\omega_k}{\partial x^j}+\frac{\partial X^j}{\partial x^k}\omega_j)dx^k=\iota_X d\omega+d(\iota_X\omega)$ 後者稱 Cartan magic formula

例 $X=F\frac{\partial}{\partial x}+G\frac{\partial}{\partial y}+H\frac{\partial}{\partial z},\omega=dv=dx\wedge dy\wedge dz$

$$L_X dx=d(Xx)=dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial z}dz$$

…

$$L_X dv = (L_X dx) \wedge dy \wedge dz + dx \wedge (L_X dy) \wedge dz + dx \wedge dy \wedge (L_X dz)$$

$$= \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\right) dv = (div \ X) dv$$