Assume that M is a compact，Riemannian and oriented。
If ξ is a Killing field and there is a smooth function $f: M \to R$ such that $\xi = gradf$，then f is constant and $\xi = 0$，prove it。

1. Killing field property

   For a Killing field ξ，$L_\xi g = 0$，this implies the Killing equation：

   $\nabla_i \xi_j + \nabla_j \xi_i = 0$，contracting indices gives $div\xi = 0$

2. Gradient field relationship
   Since $\xi = gradf$，the divergence becomes：$div(gradf) = \Delta f = 0$
   Where Δ is the Laplacian，thus，f is harmonic。
3. Harmonic function on compact manifolds
   On a compact，connected，oriented Riemannian manifold，the only harmonic functions are constants。Hence，f is constant。
4. A constant function has vanishing gradient, so $\xi = gradf = 0$

也許以下比較嚴密

ξ is a Killing vector field：$L_\omega g = 0 \Leftrightarrow \nabla_X \xi + \nabla_\xi X = 0$ for all vector fields X

$\xi = \nabla f$ where $f: M \to R$ is a smooth function，this means for any vector field X
$g(\xi, X) = df(X)$

Since $\xi = \nabla f$，we can write the Killing equation as $\nabla_X \nabla f + \nabla_{\nabla f} X = 0$

Notice that $\nabla_X \nabla f$ is the Hessian of f，$Hess(f)(X,Y) = g(\nabla_X \nabla f, Y)$

The Killing equation reduces to the condition that the Hessian of f is skew-symmetric in a certain sense. However, the Hessian of a function on a Riemannian manifold is always symmetric. The only way a symmetric tensor (the Hessian) can be skew-symmetric is if it vanishes:

$$\mathrm{Hess}(f) = 0.$$

If Hess(f)=0，then ∇f is a parallel vector field，in other words：
$\nabla\nabla f = 0 \Rightarrow \nabla \xi = 0$ This mean that ξ is a parallel vector field on M。
Now we apply a critial fact：on a compact Riemannian manifold，a nontrivial parallel gradient vector field cannot exist unless it is zero。
Why that？

Consider the norm of ξ：$|\xi|^2 = g(\xi, \xi)$

Since ξ is parallel，its norm is constant：$\nabla |\xi|^2 = 0$

If $|\xi|$ were nonzero, ξ would define a non-vanishing vector field. However, **compactness** of M combined with orientability implies topological constraints that prevent the existence of non-vanishing gradient vector fields unless they are trivial. More specifically, because $\xi = \nabla f$, the integral curves of ξ would be gradient flow lines connecting critical points of f. But if $|\xi|$ is constant and nonzero, there would be no critical points—this contradiction forces $|\xi| = 0$, so:

$$\xi = 0.$$

Since $\xi = \nabla f = 0$, it follows that f is constant (as its gradient vanishes everywhere)。