Consider the Killing vector fields on $M = S^2$ with metric 。

$K_1 = \partial_\varphi$ ， $K_2 = -\sin\varphi\partial_\theta - \cot\theta\cos\varphi\partial_\varphi$ ， $K_3 = \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi$

Verify that they satisfy the Killing equations。

$(L_X g)_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu$ ，where $\nabla_\mu X_\nu = \partial_\mu X_\nu - \Gamma^\rho_{\mu\nu} X_\rho$

$$(L_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu X^\rho + g_{\rho\mu}\partial_\nu X^\rho$$

A vector field X is a Killing vector field if the Killing equation holds：

$$(L_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu X^\rho + g_{\rho\mu}\partial_\nu X^\rho = 0$$

Means that the metric remains invariant under the flow of X。

$$\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases} ,\quad S^2 : ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$$

$$g_{\theta\theta} = 1, g_{\varphi\varphi} = \sin^2\theta, g_{\theta\varphi} = g_{\varphi\theta} = 0$$

Since $X = \frac{\partial}{\partial\varphi}$ ，we compute： $L_X g_{\theta\theta} = X(g_{\theta\theta}) + g_{\theta k}\partial_\theta X^k + g_{\theta k}\partial_\theta X^k$

Since $g_{\theta\theta} = 1$ is independent of φ ，we get $L_X g_{\theta\theta} = 0$

similarly，for $g_{\varphi\varphi} = \sin^2\theta$ ，since $X = \frac{\partial}{\partial\varphi}$ does not affect θ ，we get

$$L_X g_{\varphi\varphi} = X(g_{\varphi\varphi}) = \frac{\partial}{\partial\varphi}(\sin^2\theta) = 0$$

Thus，all components of the Lie derivative vanish，confirming that X is a Killing vector field。

Transforming back to Cartesian coordinates

$$\frac{\partial}{\partial\varphi}=\frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x}+\frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y}+\frac{\partial z}{\partial\varphi}\frac{\partial}{\partial z}=-\sin\theta\sin\varphi\frac{\partial}{\partial x}+\sin\theta\cos\varphi\frac{\partial}{\partial y}+0=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$$

$z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}$ ， $-z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}$ are Killing vector fields。

$K_2=-\sin\varphi\partial_\theta-\cot\theta\cos\varphi\partial_\varphi$ ， $K_3=\cos\varphi\partial_\theta-\cot\theta\sin\varphi\partial_\varphi$

By chain ruler：

$\partial_\theta=\cos\theta\cos\varphi\partial_x+\cos\theta\sin\varphi\partial_y-\sin\theta\partial_z$

$\partial_\varphi=-\sin\theta\sin\varphi\partial_x+\sin\theta\cos\varphi\partial_y$ 代入化簡

$K_2=-z\partial_y+y\partial_z$