For any Killing field K，prove that divK=0

1. Killing equation

   A Killing vector fieldK satisfies $\nabla_a K_b + \nabla_b K_a = 0$, which is the condition for K to generate isometries (metric-preserving transformations).

2. Contraction with metric

   Contract both sides of the Killing equation with the inverse metric g^{ab}

   $$g^{ab}(\nabla_a K_b + \nabla_b K_a) = 0$$

   Using the symmetry $g^{ab} = g^{ba}$，this simplifies to $2g^{ab}\nabla_a K_b = 0$

3. Divergence as trace

   The divergence of K is defined as $divK = \nabla_a K^a = g^{ab}\nabla_a K_b$

   From the constracted equation： $g^{ab}\nabla_a K_b \Rightarrow divK = 0$

4. **Alternative Interpretation**: The antisymmetry $\nabla_a K_b = -\nabla_b K_a$ implies $\nabla_a K_b$ is antisymmetric. Contracting a symmetric tensor (g^{ab}) with an antisymmetric tensor yields zero, confirming $\operatorname{div} K = 0$.

A vector field K is a Killing vector field if it satisfies the Killing equation：

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$$

The divergence of a vector field K is given by $div(K) = \nabla_\mu K^\mu$，this can be written as：

$$div(K) = g^{\mu\nu}\nabla_\mu K_\nu$$

From the Killing equation，we have $\nabla_\mu K_\nu = -\nabla_\nu K_\mu$

$$div(K) = g^{\mu\nu}(\nabla_\mu K_\nu) = g^{\mu\nu}(-\nabla_\nu K_\mu)$$

Since $g^{\mu\nu}$ is symmetric and $\nabla_\nu K_\mu$ is antisymetric，their contraction is zero：

$$g^{\mu\nu}\nabla_\mu K_\nu = -g^{\mu\nu}\nabla_\nu K_\mu$$

But $g^{\mu\nu}\nabla_\mu K_\nu$ is equal to $g^{\mu\nu}\nabla_\nu K_\mu$ because the indices are dummy indices and can be relabeled。Therefor $g^{\mu\nu}\nabla_\mu K_\nu = -g^{\mu\nu}\nabla_\mu K_\nu$，this implies $2g^{\mu\nu}\nabla_\mu K_\nu = 0$

Hence divK=0

Laplacian $\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$ ，在 Manifold 上 Δ 在座標變換上不順暢，因此考慮 $\Delta := div(grad)$

在 Manifold M 上 grad f 是一個向量場，用黎曼度量作內積 $< gradf(x), v >= df(v)$

Div 的定義是由 $L_X(dv) = (divX)dv$ 定義 div X，

其中 L_X 是 Lie derivative， $dv = dx^1 \wedge ... \wedge dx^n$ 是 volume element

當然，就上文 divergence 的另一種定義是 $divX = tr(\nabla X)$

在流形上 Laplacian f :=div(grad f)

Let ξ be a Killing field and assume that $\xi = gradf$ for some smooth function $f : M \to R$ Then $\nabla \xi = 0$ and $\Delta f = 0$

We have that $\langle \nabla_X \xi, Y \rangle = Hess(f)(X,Y)$ for any vector fields $X, Y \in \chi(M)$

This is skew-symmetric in X and Y since $\nabla \xi$ is skew-adjoint。

On the other hand，it is also symmetric，since torsion-free connections produce symmetric Hessian tensors。

So $\langle \nabla_X \xi, Y \rangle = 0$ for all Y implies that $\nabla_X \xi = 0$ for all X，and so $\nabla \xi = 0$

On the other hand，since Hess(f)=0，taking the trace we obtain $\Delta f = 0$ as well。