§ 共變微分(covariant derivative)

If X and Y are vector fields in Euclidean space，we can define the directional derivative $\nabla_X Y$ of Y along X。This definition，however，uses the existence of Cartesian coordinates，which no longer holds in a general manifold。

Definition

Let M be a differentiable manifold。An affine connection on M is a map $\nabla : \chi(M)\times\chi(M)\to\chi(M)$ such that

1. $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X(Y+Z)=\nabla_X Y+\nabla_X Z$
3. $\nabla_X(fY)=(X\cdot f)Y+f\nabla_X Y$

For all $X,Y,Z\in\chi(M)$ and $f,g\in C^\infty(M,R)$ (sometimes $\nabla_X Y:=\nabla(X,Y)$)

The vector field $\nabla_X Y$ is known as the covariant derivative of Y along X。

$X=\sum_i X^i\frac{\partial}{\partial x^i}, Y=\sum_i Y^i\frac{\partial}{\partial x^i}$ then $\nabla_X Y=\sum_i (XY^i+\sum_{j,k}\Gamma^i_{jk}X^jY^k)\frac{\partial}{\partial x^i}$

Where $\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}=\sum_i \Gamma^i_{jk}\frac{\partial}{\partial x^i}$

Proof

$$\nabla_X Y = \nabla_X\left(\sum_{i=1}^n Y^i\frac{\partial}{\partial x^i}\right)=\sum_{i=1}^n (X\cdot Y^i)\frac{\partial}{\partial x^i}+\sum_{i=1}^n Y^i\nabla_X\frac{\partial}{\partial x^i}$$

$$=\sum_{i=1}^n (X\cdot Y^i)\frac{\partial}{\partial x^i}+\sum_{k=1}^n Y^k\nabla_{\left(\sum_{j=1}^n X^j\frac{\partial}{\partial x^j}\right)}\frac{\partial}{\partial x^k}$$

$$=\sum_{i=1}^n (X\cdot Y^i)\frac{\partial}{\partial x^i}+\sum_{j,k=1}^n X^jY^k\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}$$

$$=\sum_{i=1}^n (X\cdot Y^i)\frac{\partial}{\partial x^i}+\sum_{i,j,k=1}^n X^jY^k\Gamma^i_{jk}\frac{\partial}{\partial x^i}$$

§ Covariant derivative of ω along X

Let ∇ be an affine connection on M。If $\omega\in\Omega^1(M)$ and $X\in\chi(M)$，we dfine the covariant derivative of ω along X by

$\nabla_X\omega(Y)=X\cdot(\omega(Y))-\omega(\nabla_X Y)$ for all $Y\in\chi(M)$

1. $\nabla_{fX+gY}\omega = f\nabla_X\omega+g\nabla_Y\omega$

2. $\nabla_X(\omega+\eta)=\nabla_X\omega+\nabla_X\eta$

3. $\nabla_X(f\omega)=(X\cdot f)\omega+f\nabla_X\omega$ for all $X,Y\in\chi(M); f,g\in C^\infty(M);\omega,\eta\in\Omega^1(M)$

證明在[RG01 p.358~360]

比較一下

$$\nabla_X Y=\sum_i (XY^i+\sum_{j,k}\Gamma^i_{jk}X^jY^k)\frac{\partial}{\partial x^i}$$

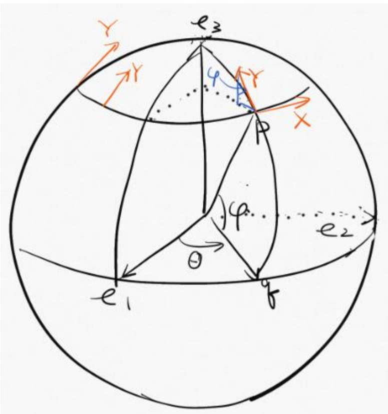
$\nabla_X\omega=\sum_i(X\omega_i-\sum_{j,k}\Gamma^k_{ji}X^j\omega_k)dx^i$ where $\omega=\sum_i\omega_i dx^i$

$$\nabla: T_pM\times\nu(M)\to T_pM$$

對 R^3 中的曲面 M, $X\in T_pM, Y\in\nu(M)$

則$\nabla_X Y=(D_X)^T$,沿 X 取向量場 Y 的微分,再取切部(在切平面T_pM 的投影。

例 半徑=1 的球面 S^2，北緯φ的小圓Γ ，e_3是北極



Y 是切於經線 指向北方的單位向量場

$p\in\Gamma, e_1,e_2,e_3$是正交標架

$p=(\cos\varphi)q+(\sin\varphi)e_3, q=(\cos\theta)e_1+(\sin\theta)e_2$ X 是小圓Γ的切向量

$Y=(-\sin\varphi)q+(\cos\varphi)e_3$

$X=\frac{\partial p}{\partial\theta}=\cos\varphi(-\sin\theta e_1+\cos\theta e_2)$

(換句話 $q=[\cos\theta,\sin\theta,0]$,

$p=[\cos\theta\cos\varphi,\sin\theta\cos\varphi,\sin\varphi]$)

$$\nabla_X Y=(D_X Y)^T=(\frac{\partial Y}{\partial\theta})^T=(-\sin\varphi\frac{dq}{d\theta})^T=(\sin\varphi\sin\theta e_1-\sin\varphi\cos\theta e_2)^T$$

設 $A=[\sin\varphi\sin\theta,-\sin\varphi\cos\theta,0]$,顯然 $A\cdot p=0$,換句話說 A 只有切部分

所以$\nabla_X Y=(\sin\varphi)(\sin\theta e_1-\cos\theta e_2)=(-\sin\varphi)\frac{X}{|X|}=(-\tan\varphi)X$

$(|X|=\cos\varphi,-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2})$