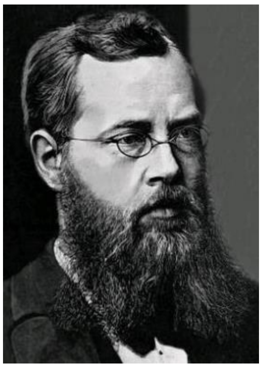
§ Lie Derivative



Sophus Lie 1842-1899

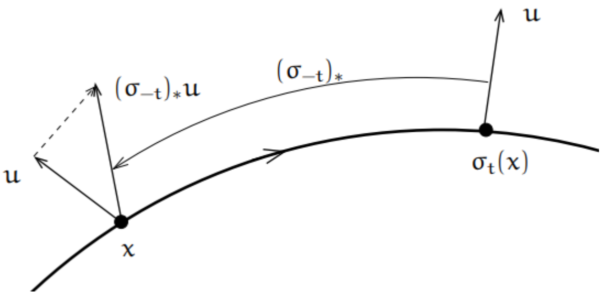
在微分幾何有三種微分：

(1)Lie derivatives $L_X Y$ (2) $\nabla_X Y$ (3) differential forms 的外微分

一個作用在流形上的張量場、向量場或函數的算子，將該張量沿著某個向量場的流做方向導數。

$$X=\sum_i X^i\frac{\partial}{\partial x^i},\ Y=\sum_i Y^i\frac{\partial}{\partial x^i}$$

φ_t is the flow a vector field X，Y is a C^∞ vector field，then the Lie derivative of Y along X is $L_X Y=\lim_{t\to 0}\frac{(\varphi_{-t})_* Y-Y}{t}=\frac{d}{dt}((\varphi_{-t})Y)\big|_{t=0}$



The dashed vector is the difference

$(\sigma_{-t})_* u-u$

$L_X Y=$

$L_X\omega=$

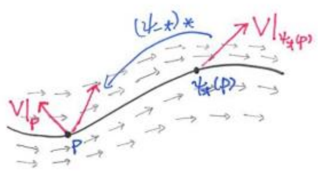
Lie derivative of vector fields. Given a vector field U on M, it associates a one-parameter family of self-diffeomorphisms ψ_t of M defined by

$$\frac{\mathrm{d}\psi_t}{\mathrm{d}t}=U\quad\left(\text{or more precisely, }\ \frac{\mathrm{d}\psi_t(p)}{\mathrm{d}t}=U|_{\psi_t(p)}\ \text{ for any } p\in M\right)$$

with ψ_0 to be the identity map. On a coordinate chart, $\psi_t=(\psi_t^1(x),\ldots,\psi_t^n(x))$ is the solution of

$$\frac{\mathrm{d}\psi_t^j(x)}{\mathrm{d}t}=u^j(\psi_t(x))\ \text{ with } \psi_0(x)=x\ . \tag{d}$$

By the fundamental theorem of O.D.E., there exists $\epsilon>0$ such that the solution exists for $t\in(-\epsilon,\epsilon)$, and is smooth in x. It follows from the uniqueness of the solution for an O.D.E. that $\psi_{t_1}\circ\psi_{t_2}=\psi_{t_1+t_2}$. Therefore, when M is compact, $\psi_t(x)$ can be defined for any $t\in\mathbb{R}$. And the inverse map of ψ_t is ψ_{-t}. It justifies the name of *one-parameter family* of *self-diffeomorphisms*.



(v) Find out the one-parameter family of diffeomorphisms generated by $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

(vi) Let ψ_t be the one-parameter family of diffeomorphisms generated by a vector field U. The *Lie derivative* of V with respect to U is defined as follows

$$(L_U V)|_p = \lim_{t\to 0} \frac{(\psi_{-t})_*(V|_{\psi_t(p)}) - V|_p}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big((\psi_{-t})_*(V|_{\psi_t(p)})\big) \tag{e}$$

where the first term in the enumerator means the push-forward of $V|_{\psi_t(p)}$ by ψ_{-t}. Since $\psi_{-t}(\psi_t(p)) = p$, $(\psi_{-t})_*(V|_{\psi_t(p)})$ is a tangent vector at p, and $(L_U V)|_p$ is the derivative of a map from $(-\epsilon,\epsilon)$ to the vector space T_pM. Show that $L_U V = [U,V]$. [*Hint*: In terms of a local coordinate, $(\psi_{-t})_*(V|_{\psi_t(p)})$ is $v^i(\psi_t(p))\frac{\partial \psi^j_{-t}}{\partial x^i}\Big|_p \frac{\partial}{\partial x^j}$. The Lie derivative (e) can be found by differentiating the coefficient functions with respect to t and evaluating at $t=0$. You shall use the defining equation (d) of ψ_t.]

Revisiting the Jacobi identity.

(vii) Let ψ_t be the one-parameter family of diffeomorphisms generated by a vector field U. Apply part (iv) and (vi) to give another proof for the Jacobi identity (b). [*Hint*: Differentiate $(\psi_{-t})_*([V,W]) = [(\psi_{-t})_*V, (\psi_{-t})_*W]$ with respect to t.]

(1) $L_X Y = [X,Y] = \sum_i (XY^i - YX^i)\frac{\partial}{\partial x^i}$

(2) $[L_X, L_Y] = L[X,Y]$

(3) $L_{aX+bY} = aL_X + bL_Y]$

(4) $L_X[Y,Z] = [L_X Y, Z] + [Y, L_X Z]$

(iii) It follows from the definition that $[U,V] = -[V,U]$. Check that the Lie bracket obeys the *Jacobi identity*, namely,

$$[U,[V,W]] + [W,[U,V]] + [V,[W,U]] = 0\ . \tag{b}$$

[*Hint*: It is a direct computation in terms of a local coordinate.]

§ Lie derivative of a form ω

$X \in \chi(M)$, $L_X\omega := \lim_{t\to 0}\frac{1}{t}(\varphi_t^*\omega - \omega) = \frac{d}{dt}(\varphi_t^*\omega)\big|_{t=0}$, Where φ_t is the local flow of X

The Lie derivative of the metric tensor g

$(L_V g)_{ij} = V^k g_{ij,k} + V^k{}_{,i} g_{kj} + V^k{}_{,j} g_{ik}$

Or $(L_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu X^\rho + g_{\rho\mu}\partial_\nu X^\rho$

The Lie derivative of forms can be expressed in terms of exterior derivative and interior product by Cartan's magic formulas：

For any $X \in \chi(M)$ and $\omega \in \Omega^p(M)$

$$L_X \omega = \iota_X d\omega + d(\iota_X \omega)$$

§ Lie derivative of tensor fields

- We want $\mathcal{L}$ obeys the following product rule w.r.t. the natural pairing between a covector field ω and a vector field Y:

$$\mathcal{L}_X(\omega(Y)) = \mathcal{L}_X\omega(Y) + \omega(\mathcal{L}_X Y).$$

Thus, it seems reasonable to define $\mathcal{L}_X\omega$ as

$$\mathcal{L}_X\omega(Y) = \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_X Y).$$

Lemma 1. *Let $\mathcal{L}$ be a Lie derivative on $\Gamma(TM)$. There is a unique Lie derivative in each tensor bundle $T^k_\ell M$, also denoted by $\mathcal{L}$, such that the following conditions are satisfied.*

(a) $\mathcal{L}_X f = Xf$, for all smooth real-valued functions f;
(b) $\mathcal{L}_X(f\sigma) = (\mathcal{L}_X f)\sigma + f\mathcal{L}_X\sigma$, for all smooth tensor filds σ, τ;
(c) $\mathcal{L}_X(\sigma \otimes \tau) = (\mathcal{L}_\sigma) \otimes \tau + \sigma \otimes \mathcal{L}_X\tau$, for smooth tensor fields σ, τ;
(d) If $Y_1, \cdots, Y_k$ are smooth vector fields and σ is a smooth k-tensor field, then

$$\mathcal{L}_X(\sigma(Y_1, \cdots, Y_k)) = (\mathcal{L}_X\sigma)(Y_1, \cdots, Y_k) + \cdots + \sigma(Y_1, \cdots, \mathcal{L}_X Y_k).$$

Lemma

Let X be a smooth vector field on smooth manifpld M，and let φ be its flow。Given a smooth covariant tensor field ω on M，the Lie derivative of ω with respect to X

$$L_X \omega = \frac{d}{dt}\Big|_{t=0} \varphi_t^* \omega = \lim_{t \to 0} \frac{\varphi_t^* \omega - \omega}{t}$$

§ covariant derivative of tensor fields

- We want D obeys the following product rule w.r.t. the natural pairing between a covector field ω and a vector field Y:

$$D_X(\omega(Y)) = D_X\omega(Y) + \omega(D_X Y).$$

Thus, it seems reasonable to define $D_X\omega$ as

$$D_X\omega(Y) = D_X(\omega(Y)) - \omega(D_X Y).$$

習作

1. 證明 $L_X Y = [X, Y]$

2. Prove that $[L_X, L_Y]$ is a derivative on the algebra $C^\infty(M)$ and

   $L_{[X,Y]} = [L_X, L_Y]$

3. $L_X Y = -L_Y X$

1. Lectures on the Geometry of Manifold Liviu Nicolaescu 1940-2023