§ Parallel transport

Parallel transport of the tangent vector V(1,1,-1) along the curve $\gamma(t)=(t,t,\frac{\pi}{4})$ from t=1 to t=0 on the helicoid $\frac{y}{x}=\tan z$ ：

To compute the parallel transport，we solve the parallel transport equation：

$$\frac{DV^i}{dt}=\frac{dV^i}{dt}+\Gamma^i_{jk}V^j\frac{dx^k}{dt}=0$$

The given curve is $\gamma(t)=(t,t,\frac{\pi}{4})$，which is a straight-line path in the surface coordinates。

The velocity vector along this path is $\dot{\gamma}(t)=(1,1,0)$

Since the given surface is the helicoid and the curve moves along a straight line in the (x,y)-plane while keeping z constant，the helicoid's connection coefficients suggest that the parallel transport will preserve the structure of the vector in the (x,y)-plane。

Because γ (t) does not change z along the path，and the helicoid's Levi-Civita connection preserves components in a simple translation along this path，the vector V remains unchanged under parallel transport。

Thus, the parallel-transported vector at t=0 remains .V(t=0)=(1,1,−1)。

Let $ds^2=\frac{1}{y^2}(dx^2+dy^2)$ be the Poincare metric on the upper half plane

$H=\{(x,y)|y>0\}$，and $\gamma=\{x^2+y^2=2|-1\le x\le 1\}$ be an arc from p(-1,1) to q(1,1)。

Let $\vec{u}=(1,0)$ be a vector at p and $\vec{v}$ be the parallel translation of $\vec{u}$ from p to q along γ。 $\vec{v}=?$

We are given a semicirclar geodesic γ

The initial tangent vector at p is $\vec{u}=(1,0)$，which is a vector in the tangent space at p。

Parallel transport in hyperbolic space preserves the inner product given by the metric。For a geodesic that is a semicircle centered on the x-axis，parallel transport along it corresponds to a **rotation** by the angle subtended by the arc。

1. The semicircle $x^2 + y^2 = 2$ is centered at the origin and passes through (−1,1) and (1,1)
2. The angle subtended by this semicircle from p to q at the origin is π (since it forms a half-circle)。
3. Parallel transport in this case corresponds to a rotation by π around the origin。

Thus, if we start with $\vec{u} = (1,0)$ at p(−1,1)，the parallel transport of this vector along the geodesic to q(1,1) will result in：$\vec{v} = (-1,0)$

The inner product in the Poincare metric is $\langle \vec{u}, \vec{v} \rangle = \frac{1}{y^2}(u_1 v_1 + u_2 v_2)$

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Answer by DeepSeek

To solve the problem of parallel transporting the vector $u = (1, 0)$ at the point $p(-1, 1)$ along the geodesic $x^2 + y^2 = 2$ to the point $q(1, 1)$ in the Poincaré upper half-plane with the metric $\frac{dx^2+dy^2}{y^2}$, we can use the angle preservation property of parallel transport along a geodesic.

1. **Identify the Geodesic and Tangent Vectors**:
   - The given path $x^2 + y^2 = 2$ is a geodesic in the Poincaré upper half-plane.
   - The tangent vector at $p(-1, 1)$ is calculated as $(1, 1)$.
   - The tangent vector at $q(1, 1)$ is calculated as $(1, -1)$.

2. **Initial Angle Calculation**:
   - The initial vector $u = (1, 0)$ at $p(-1, 1)$ forms an angle of $\frac{\pi}{4}$ with the tangent vector $(1, 1)$.

3. **Angle Preservation**:
   - Parallel transport along a geodesic preserves the angle between the vector and the tangent to the geodesic. Therefore, the transported vector v at $q(1, 1)$ must form the same angle $\frac{\pi}{4}$ with the tangent vector $(1, -1)$.

4. **Determine the Transported Vector**:
   - The vector v at $q(1, 1)$ must have the same magnitude as u (which is 1) and form an angle of $\frac{\pi}{4}$ with the tangent vector $(1, -1)$.
   - By symmetry and the conformal nature of the Poincaré metric, the transported vector v that maintains the angle $\frac{\pi}{4}$ and points in the direction consistent with the original vector u is found to be $(1, 0)$.