§ Affine connection $\nabla_X Y$

Covariant derivative of Y along X

$X=\sum_i X^i \frac{\partial}{\partial x^i}$ 、 $Y=\sum_i Y^i \frac{\partial}{\partial x^i}$ then $\nabla_X Y=\sum_i (XY^i+\sum_{j,k}\Gamma^i_{jk}X^j Y^k)\frac{\partial}{\partial x^i}$

Covariant derivative of ω along X

$\nabla_X \omega=\sum_i (X\omega^i-\sum_{jk}\Gamma^k_{ji}X^j\omega_k)dx^i$

In classical Differential Geometry $X_{ij}=\Gamma^k_{ij}X_k+b_{ij}N$

And $\Gamma^i_{jk}=\frac{1}{2}\sum_i g^{il}(\frac{\partial g_{kl}}{\partial x^j}+\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}), g^{ij}=(g_{ij})^{-1}$

1. $\nabla_{fX+gY}Z=f\nabla_X Z+g\nabla_Y Z$
2. $\nabla_X(Y+Z)=\nabla_X Y+\nabla_X Z$
3. $\nabla_X(fY)=(X\cdot f)Y+f\nabla_X Y$

EX

$\frac{d}{dt}<V,W>=<\frac{DV}{dt},W>+<V,\frac{DW}{dt}>$

相容 $X\cdot<Y,Z>=<\nabla_X Y,Z>+<Y,\nabla_X Z>$

對稱 $\nabla_X Y-\nabla_Y X=[X,Y]$(called torsion free)

When ∇ compatible with metric and torsion free then ∇ is called Levi-Civita connection(Riemannian connection)。

黎曼流形(M,<,>) with an affine connection ∇

∇與<,>相容$\Leftrightarrow$ C 是 smooth curve，X,Y 是沿 c 的平行向量場 則<X,Y>=constant

A Levi-Civita connection preserves length and angles under parallel transport。

Proof

Let $T=\alpha'(t)$ be tangent to curve $\alpha(t)$

X , Y be parallel transported along α

$\nabla_T X=\nabla_T Y=0$

$\nabla_T<X,X>=<\nabla_T X,X>+<X,\nabla_T X>=0$，$\therefore\|X\|$ is constant。

$\nabla_T<X,Y>=<\nabla_T X,Y>+<X,\nabla_T Y>=0$，$\therefore<X,Y>$ is constant。

$\cos\theta = \frac{<X,Y>}{\|X\|\|Y\|}$=constant。

定理[DG06] p.135

(M ,g) is a Riemann manifold，Then there exists a unique symmetric connection ∇ on TM compatible with the metric g i.e. $T(\nabla)=0, \nabla g = 0$。

The connection ∇ is usually called the Levi-Civita connection associated to the metric g。

Proof. Uniqueness. We will achieve this by producing an *explicit* description of a connection with the above two mproperties.

Let ∇ be such a connection, i.e.,

$$\nabla g = 0 \text{ and } \nabla_X Y - \nabla_Y X = [X,Y], \quad \forall X,Y \in \text{Vect}(M).$$

For any $X, Y, Z \in \text{Vect}(M)$ we have

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

since $\nabla g = 0$. Using the symmetry of the connection we compute

$$Zg(X,Y) - Yg(Z,X) + Xg(Y,Z) = g(\nabla_Z X, Y) - g(\nabla_Y Z, X) + g(\nabla_X Y, Z)$$

$$+g(X, \nabla_Z Y) - g(Z, \nabla_Y X) + g(Y, \nabla_X Z)$$
$$= g([Z,Y],X) + g([X,Y],Z) + g([Z,X],Y) + 2g(\nabla_X Z, Y).$$

We conclude that

$$g(\nabla_X Z, Y) = \frac{1}{2}\{Xg(Y,Z) - Yg(Z,X) + Zg(X,Y)$$

$$-g([X,Y],Z) + g([Y,Z],X) - g([Z,X],Y)\}. \tag{4.1.3}$$

The above equality establishes the uniqueness of ∇.

Using local coordinates $(x^1, \ldots, x^n)$ on M we deduce from (4.1.3), with $X = \partial_i = \frac{\partial}{\partial x_i}$, $Y = \partial_k = \frac{\partial}{\partial x_k}$, $Z = \partial_j = \frac{\partial}{\partial x_j}$), that

$$g(\nabla_i \partial_j, \partial_k) = g_{k\ell}\Gamma_{ij}^{\ell} = \frac{1}{2}\left(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\right).$$

Above, the scalars Γ_{ij}^{ℓ} denote the *Christoffel symbols* of ∇ in these coordinates, i.e.,

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^{\ell}\partial_\ell.$$

If $(g^{i\ell})$ denotes the inverse of $(g_{i\ell})$ we deduce

$$\Gamma_{ij}^{\ell} = \frac{1}{2} g^{k\ell}\left(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\right). \tag{4.1.4}$$

Existence. It boils down to showing that (4.1.3) indeed defines a connection with the required properties. The routine details are left to the reader. □

Proposition 4.1.44. $\nabla_X dV_g = 0$, $\forall X \in \text{Vect}\,(M)$.

Proof. We have to show that for any $p \in M$

$$(\nabla_X dV_g)(e_1, ..., e_n) = 0, \tag{4.1.13}$$

where $e_1, ..., e_p$ is a basis of T_pM. Choose normal coordinates $(\boldsymbol{x}^i)$ near p. Set $\partial_i = \frac{\partial}{\partial \boldsymbol{x}^i}$, $\boldsymbol{g}_{ij} = g(\partial_i, \partial_k)$, and $e_i = \partial_i \,|_p$. Since the expression in (4.1.13) is linear in X, we may as well assume $X = \partial_k$, for some $k = 1, ..., n$. We compute

$$\begin{aligned}(\nabla_X dV_g)(e_1, ..., e_n) = X(dV_g(\partial_1, ..., \partial_n))\,|_p \\ - \sum_i dv_g(e_1, ..., (\nabla_X \partial_i)\,|_p, ..., \partial_n).\end{aligned} \tag{4.1.14}$$

We consider each term separately. Note first that $dV_g(\partial_1, ..., \partial_n) = (\det(\boldsymbol{g}_{ij}))^{1/2}$, so that

$$X(\det(\boldsymbol{g}_{ij}))^{1/2}\,|_p = \partial_k(\det(\boldsymbol{g}_{ij}))^{1/2}\,|_p$$

is a linear combination of products in which each product has a factor of the form $\partial_k \boldsymbol{g}_{ij}\,|_p$. Such a factor is zero since we are working in normal coordinates. Thus, the first term in(4.1.14) is zero. The other terms are zero as well since in normal coordinates at p we have the equality

$$\nabla_X \partial_i = \nabla_{\partial_k} \partial_i = 0.$$

Proposition 4.1.44 is proved. □