§ Structure Equation

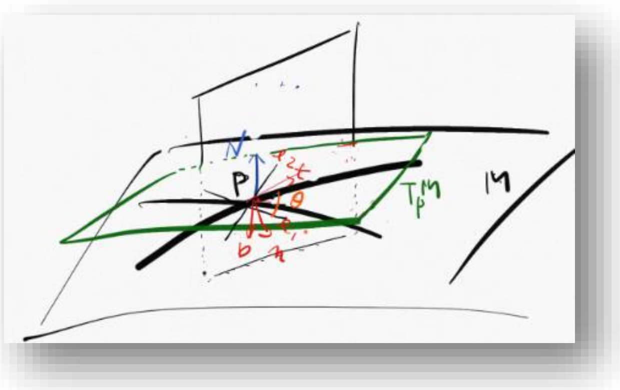
活動標架法(微分幾何講稿 p.28)

以下 $\overline{\xi_1}=T,\overline{\xi_3}=N$ ，這裡取 $\vec{\xi}_2=\vec{\xi}_3\times\vec{\xi}_1$ 書中取 $B=T\times N$ ，差一個負號。

$T'=\kappa N,\tau=<N',B>$ 是曲率與扭率。而有 Frenet 公式

$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix}=\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$ 當 Lie group 為正交變換時，其相應的 Lie algebra 為反對稱方陣環。

Moving frames on surface theory



$$\vec{\xi}_1=\frac{X_u}{|X_u|}，\vec{\xi}_3=\frac{X_u\times X_v}{|X_u\times X_v|}，$$

$$\vec{\xi}_2=\vec{\xi}_3\times\vec{\xi}_1$$

$$A=(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3)=\begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 \end{pmatrix}$$

$$X=\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},\hat{\theta}=\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix}$$

$$\omega=A^t dA=\begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^0 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix}=\left(\omega_j^i\right)=(<\overline{\xi_i},d\overline{\xi_j}>)$$ ，called matrix of connection form

With $\eta^0=<\vec{\xi}_2,d\vec{\xi}_1>,\eta^1=<\vec{\xi}_3,d\vec{\xi}_1>,\eta^2=<\vec{\xi}_3,d\vec{\xi}_2>$

即 $\omega_1^2=-B\cdot dT,\omega_1^3=-N\cdot dT,\xi_2^3=-N\cdot dB$

$\hat{\theta}=A^t dX$ ，then $dX=A\hat{\theta},dA=A\omega$

(then $\theta^i=\overline{\xi_i}\cdot dX$ ，即 $\theta^1=<dX,T>,\theta^2=<\mathrm{dX,B}>,\theta^3=<dX,N>$

由 d(d X)=0 推出 $d\hat{\theta}=-\omega\wedge\hat{\theta}\Rightarrow d\theta^1=\eta^0\wedge\theta^2,d\theta^2=-\eta^0\wedge\theta^1,\eta^1\wedge\theta^1+\eta^2\wedge\theta^2=0$

由 d(d A)=0 推出 $d\omega = -\omega \wedge \omega \Rightarrow \begin{cases} d\eta^0 = -\eta^1 \wedge \eta^2 \\ d\eta^1 = -\eta^2 \wedge \eta^0 \\ d\eta^2 = -\eta^0 \wedge \eta^1 \end{cases}$

$$0 = d(dX) = d(A\hat{\theta}) = dA \wedge \hat{\theta} + Ad\hat{\theta}$$

$$= (A\omega) \wedge \hat{\theta} + Ad\hat{\theta}$$ So $d\hat{\theta} = -\omega \wedge \hat{\theta}$

$$= A(\omega \wedge \hat{\theta} + d\hat{\theta})$$

同理 由 d(d A)=0 推出 $d\omega = -\omega \wedge \omega$

Let X=X(u,v) 位置向量

$$dX = X_u du + X_v dv$$

$$\theta^1 = <\vec{\xi_1}, dX> = <\vec{\xi_1}, X_u> du + <\vec{\xi_1}, X_v> dv$$

$$\theta^2 = <\vec{\xi_2}, dX> = <\vec{\xi_2}, X_u> du + <\vec{\xi_2}, X_v> dv$$

$$\theta^3 = <\vec{\xi_3}, dX> = 0$$

$$d\hat{\theta} = -\begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^1 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \\ 0 \end{pmatrix}$$

$$d\theta^1 = \eta^0 \wedge \theta^2$$

$$d\theta^2 = -\eta^0 \wedge \theta^1$$

$d\theta^3 = -\eta^1 \wedge \theta^1 - \eta^2 \wedge \theta^2 = 0$，所以 $\eta^1 \wedge \theta^1 + \eta^2 \wedge \theta^2 = 0$

$$d\omega = -\omega \wedge \omega = -\begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^0 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^1 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix}$$，then

$d\eta^0 = -\eta^1 \wedge \eta^2$ Gauss equation

$\begin{cases} d\eta^1 = -\eta^2 \wedge \eta^0 \\ d\eta^2 = -\eta^0 \wedge \eta^1 \end{cases}$ Codazzi-Mainardi equation

Definition　　identities　　structure equation

$\hat{\theta} = A^t dX$　　$A^T A =$　　$d\hat{\theta} = -\omega \wedge$

$\omega = A^t dA$　　$\omega + \omega^T = 0$　　$d\omega = -\omega \wedge$

$dX = X_u du + X_v dv = \theta^1 \vec{\xi}_1 + \theta^2 \vec{\xi}_2$ derives the Area form $\theta^1 \wedge \theta^2 = \left| X_u \times X_v \right| dudv$

切平面為$\overline{\xi_1}(=T), \overline{\xi_2}(=B)$所張，$\theta^1, \theta^2$分別是 dX 在 T，B 的分量。

First fundamental form $I = \langle dX, dX \rangle = Edu^2 + 2Fdudv + Gdv^2$

Second fundamental form $II = -\langle dX, dN \rangle = Ldu^2 + 2\mathrm{Mdudv} + \mathrm{Ndv}^2$

Definition $\eta^1 \wedge \eta^2 = K(\theta^1 \wedge \theta^2)$

$-\theta^2 \wedge \eta^1 + \theta^1 \wedge \eta^2 = 2H(\theta^1 \wedge$

Because $\theta^1 \wedge \theta^2 \neq 0$，$\theta^1, \theta^2$ form the basis of 1-form at every point on M

Assume $\eta^1 = a\theta^1 + b\theta^2, \eta^2 = c\theta^1 + d\theta^2$

For $\eta^1 \wedge \theta^1 + \eta^2 \wedge \theta^2 = 0$ then b=c

Rewrite let $\eta^1 = a\theta^1 + c\theta^2, \eta^2 = c\theta^1 + b\theta^2$

$$\begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = M \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}$$

$K = \det M = \kappa_1 \kappa_2 = ab - c^2$ for which κ_1, κ_2 are eigenvalues of M

Because determinant and trace are invariance of 對角化

$$H = trM = \frac{\kappa_1 + \kappa_2}{2}$$

Classical differential geometry

$\dfrac{dT}{ds} = \kappa_n N + \kappa_g Y$，$\kappa_n = \dfrac{II}{I}$ with $II = -dX \cdot dN = edu^2 + 2fdudv + gdv^2$

$e = X_{uu} \cdot N, f = X_{uv} \cdot N, g = X_{vv} \cdot N$

$H = \dfrac{Eg - 2Ff + Ge}{2(EG - F^2)}$，$K = \dfrac{eg - f^2}{EG - F^2}$

Example

1. $S^2 : X = (a\cos u\sin v, a\sin u\sin v, a\cos v)$, compute the Gauss curvature K

$$X_u = (-a\sin\cos v, -a\sin\sin v, a\cos u)$$

$$X_v = (-a\cos u\sin v, a\cos u\cos v, 0)$$

$$X_u \times X_v = (-a^2\cos^2 u\cos v, -a^2\cos^2 u\sin v, -a^2\sin u\cos u)$$

$$\vec{\xi_1} = X_u / |X_u| = (-\sin u\cos v, -\sin u\sin v, \cos u)$$

$$\vec{\xi_3} = X_u \times X_v \Big/ |X_u \times X_v| = (-cou\cos v, -\cos u\sin v, -\sin u)$$

$$\vec{\xi_2} = \vec{\xi_3} \times \vec{\xi_1} = (-\sin v, \cos v, 0)$$

$$\omega = \begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^0 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sin u dv & -du \\ du & 0 & -\cos dv \\ du & \cos u dv & 0 \end{pmatrix} = A^T dA$$

$$\eta^0 = \omega_1^2 = < \vec{\xi_2}, \vec{\xi_1} > = -\sin u dv \text{ , } \eta^1 = \omega_1^3 = < \vec{\xi_3}, d\vec{\xi_1} > = du \text{ ,}$$

$$\eta^2 = \omega_2^3 = < \vec{\xi_3}, d\vec{\xi_2} > = \cos u dv$$

$$\theta^1 = < \vec{\xi_1}, dX > = a\sin v du, \theta^2 = < \vec{\xi_2}, dX > = a dv$$

$$\theta^1 \wedge \theta^2 = |X_u \times X_v| du dv = a^2 \cos u du$$

$$\eta^1 = \sin v du, \eta^2 = dv \text{ , } \eta^1 \wedge \eta^2 = \cos u du$$

$$K = \frac{\eta^1 \wedge \eta^2}{\theta^1 \wedge \theta^2} = \frac{1}{a^2}, H = \frac{-\theta^2 \wedge \eta^1 + \theta^1 \wedge \eta^2}{2\theta^1 \wedge \theta^2} = \frac{1}{a}$$

2. Hyperbolic plane $g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$

Compute the Gauss Curvature K

$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}$ Then $< X_1, X_1 > = \frac{1}{y^2}, < X_1, X_2 > = 0, < X_2, X_2 > = \frac{1}{y^2}$

Take $E_1 = yX_1, E_2 = yX_2$ form an orthonormal frames

Then $\omega^1 = \frac{1}{y} dx, \omega^2 = \frac{1}{y} dy$

$$d\omega^1 = (-\frac{1}{y^2}) dy \wedge dx = \frac{1}{y^2} dx \wedge dy = \omega^1 \wedge \omega^2$$

$d\omega^2 = 0$

Since $d\omega^i = \sum_j \omega^j \wedge \omega_j^i$

$d\omega^1 = \omega^1 \wedge \omega_1^1 + \omega^2 \wedge \omega_2^1 = \omega^2 \wedge \omega_2^1 = \omega_1^2 \wedge \omega^2$

$d\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^2 \wedge \omega_2^2 = \omega^1 \wedge \omega_1^2 = 0$

Let $\omega_1^2 = a\omega^1 + b\omega^2$ ， $\omega^1 \wedge \omega_1^2 = 0$ $\therefore$ b=0

$\omega_1^2 \wedge \omega^2 = d\omega^1 = \omega^1 \wedge \omega^2 = a\omega^1 \wedge \omega^2$ $\therefore$ a=1

$d\omega_1^2 = d\omega^1 = \omega^1 \wedge \omega^2 = -K\omega^1 \wedge \omega^2$ $\therefore$ K=-1

$$\omega = \begin{pmatrix} 0 & -\frac{1}{y} \\ \frac{1}{y} & 0 \end{pmatrix} dx \text{ ， } d\theta = -\omega \wedge \theta \text{ ， } d\omega + \omega \wedge \omega = \Omega$$

$$\Omega = d\omega = \begin{pmatrix} 0 & \frac{1}{y^2} \\ -\frac{1}{y^2} & 0 \end{pmatrix} dy \wedge dx = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \theta^1 \wedge \theta^2$$

The Gauss curvature is $K = \frac{1}{|g|} g(\Omega(\partial_x, \partial_y)\partial_y, \partial_x) = y^4(-\frac{1}{y^4}) = -1$

3. Helicoid (螺旋曲面)

$X(u,v) = (u\cos v, u\sin v, v)$, $p = \sqrt{1+u^2}$

$X_u =$

$X_v =$

$\vec{\xi_1} =$

$\vec{\xi_3} =$

$\vec{\xi_2} = \vec{\xi_3} \times \vec{\xi_1} =$

$\theta^1 = <\vec{\xi_1}, X_u > du + <\vec{\xi_1}, X_v > dv = du$ ， $\theta^2 = <\vec{\xi_2}, X_u > du + <\vec{\xi_2}, X_v > dv = pdv$

$\eta^1 = < \vec{\xi_3}, d\vec{\xi_1} > = -\frac{1}{p} dv$ ， $\eta^2 = < \vec{\xi_3}, d\vec{\xi_2} > = -\frac{1}{p^2} du$

Then H=0， $K = -\frac{1}{p^4} = -\frac{1}{(1+u^2)^2}$

4. …

[大域微分幾何]p.67 球面 S^n 的曲率

截曲率 $R_{ijij} = \frac{1}{r^2}, \forall i, j, i \neq j$ 當 n=2 時 $R_{1212} = \frac{1}{r^2}$ 就是 Gauss 曲率。

p.63 § 3 計算 Clifford 環面的曲率

[大域微分幾何] p.346 結構方程在曲面論的應用

Exercise 2.8

(1) Let $\{X_1, \ldots, X_n\}$ be a field of frames on an open set V of a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ with Levi–Civita connection ∇. The associated **structure functions** C_{ij}^k are defined by

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ij}^k X_k.$$

Show that:

(a) $C_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i$;

(b) $\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^n g^{il} \left(X_j \cdot g_{kl} + X_k \cdot g_{jl} - X_l \cdot g_{jk} \right) + \frac{1}{2} C_{jk}^i - \frac{1}{2} \sum_{l,m=1}^n g^{il} \left(g_{jm} C_{kl}^m + g_{km} C_{jl}^m \right)$;

(c) $d\omega^i + \frac{1}{2} \sum_{j,k=1}^n C_{jk}^i \omega^j \wedge \omega^k = 0$, where $\{\omega^1, \ldots, \omega^n\}$ is the field of dual coframes.

(2) Let $\{X_1, \ldots, X_n\}$ be a field of frames on an open set V of a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$. Show that a connection ∇ on M is compatible with the metric on V if and only if

$$X_k \cdot \langle X_i, X_j \rangle = \langle \nabla_{X_k} X_i, X_j \rangle + \langle X_i, \nabla_{X_k} X_j \rangle$$

for all i, j, k.

(4) Determine all surfaces of revolution with constant Gauss curvature.

(5) Let M be the image of the parameterization $\varphi : (0, +\infty) \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (u \cos v, u \sin v, v),$$

and let N be the image of the parameterization $\psi : (0, +\infty) \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\psi(u, v) = (u \cos v, u \sin v, \log u).$$

Consider in both M and N the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^3$. Show that the map $f : M \to N$ defined by

$$f(\varphi(u, v)) = \psi(u, v)$$

preserves the Gauss curvature but is not a local isometry.

(6) Consider the metric

$$g = A^2(r)dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2 \theta \, d\varphi \otimes d\varphi$$

on $M = I \times S^2$, where r is a local coordinate on $I \subset \mathbb{R}$ and (θ, φ) are spherical local coordinates on S^2.

(a) Compute the Ricci tensor and the scalar curvature of this metric.

(b) What happens when $A(r) = (1 - r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to S^3)?

(c) And when $A(r) = (1 + r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to the **hyperbolic** 3-**space**)?

(d) For which functions $A(r)$ is the scalar curvature constant?

(7) Let M be an oriented Riemannian 2-manifold and let p be a point in M. Let D be a neighborhood of p in M homeomorphic to a disc, with a smooth boundary ∂D. Consider a point $q \in \partial D$ and a unit vector $X_q \in T_q M$. Let X be the parallel transport of X_q along ∂D in the positive direction. When X returns to q it makes an angle $\Delta\theta$ with the initial vector X_q. Using fields of positively oriented orthonormal frames $\{E_1, E_2\}$ and $\{F_1, F_2\}$ such that $F_1 = X$, show that

$$\Delta\theta = \int_D K.$$

Conclude that the Gauss curvature of M at p satisfies

$$K(p) = \lim_{D \to p} \frac{\Delta\theta}{\mathrm{vol}(D)}.$$

(8) Compute the geodesic curvature of a positively oriented circle on:

(a) $\mathbb{R}^2$ with the Euclidean metric and the usual orientation;
(b) S^2 with the usual metric and orientation.

(9) Let c be a smooth curve on an oriented 2-manifold M as in the definition of geodesic curvature. Let X be a vector field parallel along c and let θ be the angle between X and $\dot{c}(s)$ along c in the given orientation. Show that the geodesic curvature of c, k_g, is equal to $\frac{d\theta}{ds}$. (**Hint:** Consider two fields of orthonormal frames $\{E_1, E_2\}$ and $\{F_1, F_2\}$ positively oriented such that $E_1 = \frac{X}{\|X\|}$ and $F_1 = \dot{c}$).