§ Cartan Structure Equation [N3202]

§ 活動標架法(微分幾何講稿 p.28)

以下$\overline{\xi_1}=T,\overline{\xi_3}=N$，這裡取$\vec{\xi_2}=\vec{\xi_3}\times\vec{\xi_1}$ 書中取$B=T\times N$，差一個負號。

$T'=\kappa N,\tau=<N',B>$是曲率與扭率。而有 Frenet 公式

$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix}=\begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$ 當 Lie group 為正交變換時，其相應的 Lie algebra 為反對稱方陣環。

Moving frames on surface theory

$\vec{\xi}_1=\dfrac{X_u}{|X_u|}$，$\vec{\xi_3}=\dfrac{X_u\times X_v}{|X_u\times X_v|}$，$\vec{\xi_2}=\vec{\xi_3}\times\vec{\xi_1}$

$$A=(\vec{\xi_1},\vec{\xi_2},\vec{\xi_3})=\begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 \end{pmatrix}\quad X=\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix},\hat{\theta}=\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix}$$

$\omega=A^t dA=\begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^0 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix}=\left(\omega_j^i\right)=(<\overline{\xi_i},d\overline{\xi_j}>)$，called matrix of connection form

With $\eta^0=<\vec{\xi_2},d\vec{\xi_1}>,\eta^1=<\vec{\xi_3},d\vec{\xi_1}>,\eta^2=<\vec{\xi_3},d\vec{\xi_2}>$

即$\omega_1^2=-B\cdot dT,\omega_1^3=-N\cdot dT,\xi_2^3=-N\cdot dB$

$\hat{\theta}=A^t dX$，then $dX=A\hat{\theta},dA=A\omega$

(then $\theta^i=\vec{\xi_i}\cdot dX$，即$\theta^1=<dX,T>,\theta^2=<\mathrm{dX,B}>,\theta^3=<dX,N>$

由 d(d X)=0 推出$d\hat{\theta}=-\omega\wedge\hat{\theta}\Rightarrow d\theta^1=\eta^0\wedge\theta^2,d\theta^2=-\eta^0\wedge\theta^1,\eta^1\wedge\theta^1+\eta^2\wedge\theta^2=0$

由 d(d A)=0 推出$d\omega=-\omega\wedge\omega\Rightarrow\begin{cases} d\eta^0=-\eta^1\wedge\eta^2 \\ d\eta^1=-\eta^2\wedge\eta^0 \\ d\eta^2=-\eta^0\wedge\eta^1 \end{cases}$

$0 = d(dX) = d(A\hat{\theta}) = dA \wedge \hat{\theta} + Ad\,\hat{\theta}$ ，So $d\,\hat{\theta} = -\omega \wedge \hat{\theta}$

$= (A\omega) \wedge \hat{\theta} + Ad\,\hat{\theta} = A(\omega \wedge \hat{\theta} + d\,\hat{\theta})$

同理 由 d(d A)=0 推出 $d\omega = -\omega \wedge \omega$

Let X=X(u,v) 位置向量 $dX = X_u du + X_v dv$

$\theta^1 = <\vec{\xi_1}, dX> = <\vec{\xi_1}, X_u> du + <\vec{\xi_1}, X_v> dv$

$\theta^2 = <\vec{\xi_2}, dX> = <\vec{\xi_2}, X_u> du + <\vec{\xi_2}, X_v> dv$

$\theta^3 = <\vec{\xi_3}, dX> = 0$

$$d\,\hat{\theta} = -\begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^1 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix}\begin{pmatrix} \theta^1 \\ \theta^2 \\ 0 \end{pmatrix}$$

$d\theta^1 = \eta^0 \wedge \theta^2 \quad d\theta^2 = -\eta^0 \wedge \theta^1$

$d\theta^3 = -\eta^1 \wedge \theta^1 - \eta^2 \wedge \theta^2 = 0$ ，所以 $\eta^1 \wedge \theta^1 + \eta^2 \wedge \theta^2 = 0$

$$d\omega = -\omega \wedge \omega = -\begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^0 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^1 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix}$$ ，then

$d\eta^0 = -\eta^1 \wedge \eta^2$ Gauss equation

$\begin{cases} d\eta^1 = -\eta^2 \wedge \eta^0 \\ d\eta^2 = -\eta^0 \wedge \eta^1 \end{cases}$ Codazzi-Mainardi equation

Definition	identities	structure equation
$\hat{\theta} = A^t dX$	$A^T A=$ I	$d\,\hat{\theta} = -\omega\wedge\hat{\theta}$
$\omega = A^t dA$	$\omega + \omega^T = 0$	$d\omega = -\omega \wedge \omega$

$dX = X_u du + X_v dv = \theta^1 \vec{\xi_1} + \theta^2 \vec{\xi_2}$ derives the Area form $\theta^1 \wedge \theta^2 = |X_u \times X_v| dudv$

切平面為 $\overline{\xi_1}(=T), \overline{\xi_2}(=B)$ 所張，θ^1, θ^2 分別是 dX 在 T，B 的分量。

First fundamental form $I=\langle dX,dX\rangle = Edu^2+2Fdudv+Gdv^2$

Second fundamental form $II=-\langle dX,dN\rangle = Ldu^2$+2Mdudv+Ndv^2

Definition $\eta^1\wedge\eta^2 = K(\theta^1\wedge\theta^2)$

$$-\theta^2\wedge\eta^1+\theta^1\wedge\eta^2 = 2H(\theta^1\wedge\theta$$

Because $\theta^1\wedge\theta^2\neq 0$ ， θ^1,θ^2 form the basis of 1-form at every point on M

Assume $\eta^1=a\theta^1+b\theta^2,\eta^2=c\theta^1+d\theta^2$

For $\eta^1\wedge\theta^1+\eta^2\wedge\theta^2=0$ then b=c

Rewrite let $\eta^1=a\theta^1+c\theta^2,\eta^2=c\theta^1+b\theta^2$

$$\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}=\begin{pmatrix}a & c\\ c & b\end{pmatrix}\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix}=M\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix}$$

$K=\det M=\kappa_1\kappa_2=ab-c^2$ for which κ_1,κ_2 are eigenvalues of M

Because determinant and trace are invariance of 對角化

$$H=trM=\frac{\kappa_1+\kappa_2}{2}$$

Example

1. $S^2: X=(a\cos u\sin v, a\sin u\sin v, a\cos v)$ ，compute the Gauss curvature K

$$X_u=(-a\sin\cos v,-a\sin\sin v,a\cos u)$$

$$X_v=(-a\cos u\sin v,a\cos u\cos v,0)$$

$$X_u\times X_v=(-a^2\cos^2 u\cos v,-a^2\cos^2 u\sin v,-a^2\sin u\cos u)$$

$$\vec{\xi}_1=X_u/|X_u|=(-\sin u\cos v,-\sin u\sin v,\cos u)$$

$$\vec{\xi}_3 = X_u \times X_v \Big/ \left|X_u \times X_v\right| = (-cou\cos v, -\cos u \sin v, -\sin u)$$

$$\vec{\xi}_2 = \vec{\xi}_3 \times \vec{\xi}_1 = (-\sin v, \cos v, 0)$$

$$\omega = \begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^0 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sin u dv & -du \\ du & 0 & -\cos dv \\ du & \cos u dv & 0 \end{pmatrix} = A^T dA$$

$\eta^0 = \omega_1^2 =< \vec{\xi}_2, \vec{\xi}_1 >= -\sin u dv$ ， $\eta^1 = \omega_1^3 =< \vec{\xi}_3, d\vec{\xi}_1 >= du$ ，

$\eta^2 = \omega_2^3 =< \vec{\xi}_3, d\vec{\xi}_2 >= \cos u dv$

$\theta^1 =< \vec{\xi}_1, dX >= a\sin v du, \theta^2 =< \vec{\xi}_2, dX >= adv$

$\theta^1 \wedge \theta^2 = \left|X_u \times X_v\right| du dv = a^2 \cos u du$

$\eta^1 = \sin v du, \eta^2 = dv$ ， $\eta^1 \wedge \eta^2 = \cos u du$

$$K = \frac{\eta^1 \wedge \eta^2}{\theta^1 \wedge \theta^2} = \frac{1}{a^2}, H = \frac{-\theta^2 \wedge \eta^1 + \theta^1 \wedge \eta^2}{2\theta^1 \wedge \theta^2} = \frac{1}{a}$$

2. Helicoid (螺旋曲面)

$X(u,v) = (u\cos v, u\sin v, v)$, $p = \sqrt{1+u^2}$

$X_u =$ $X_v =$ $\vec{\xi}_1 =$ $\vec{\xi}_3 =$ $\vec{\xi}_2 = \vec{\xi}_3 \times \vec{\xi}_1 =$

$\theta^1 =< \vec{\xi}_1, X_u > du + < \vec{\xi}_1, X_v > dv = du$ ， $\theta^2 =< \vec{\xi}_2, X_u > du + < \vec{\xi}_2, X_v > dv = pdv$

$\eta^1 =< \vec{\xi}_3, d\vec{\xi}_1 >= -\frac{1}{p} dv$ ， $\eta^2 =< \vec{\xi}_3, d\vec{\xi}_2 >= -\frac{1}{p^2} du$

Then H=0， $K = -\frac{1}{p^4} = -\frac{1}{(1+u^2)^2}$

[大域微分幾何]p.67 球面 S^n 的曲率

截曲率 $R_{ijij} = \frac{1}{r^2}, \forall i, j, i \neq j$ 當 n=2 時 $R_{1212} = \frac{1}{r^2}$ 就是 Gauss 曲率。

p.63 § 3 計算 Clifford 環面的曲率

[大域微分幾何] p.346 結構方程在曲面論的應用
Cartan structure equation 的幾何意義為何，有此一說：

1. $d\omega^j = \omega^i \wedge \omega_i^j + \tau^j \Leftrightarrow \nabla_X Y - \nabla_Y X - [X,Y] = \tau(X,Y)$

$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j \Leftrightarrow [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z = R(X,Y)Z$

§ Cartan structure equation

$d\omega^i = \sum_j \omega^j \wedge \omega_j^i$ ， $\omega_i^j + \omega_j^i = 0$ ， $\Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$

曲率形式Ω_j^i與黎曼曲率張量R_{jkl}^i的關係為$\Omega_j^i = \dfrac{1}{2} R_{jkl}^i \omega^k \wedge \omega^l$ 。在二維曲面上

$\Omega_2^1 = K\omega^1 \wedge \omega^2$ ，對應的張量$R_{212}^1 =$ K

$\Omega_2^1 = \dfrac{1}{2}(R_{212}^1 \omega^1 \wedge \omega^2 + R_{221}^1 \omega^2 \wedge \omega^1)$

1. Hyperbolic plane $g = \dfrac{1}{y^2}(dx^2 + dy^2)$ Compute the Gauss Curvature K

$\omega^1 = \dfrac{1}{y}dx, \omega^2 = \dfrac{1}{y}dy$

$d\omega^1 = \dfrac{-1}{y^2}dy \wedge dx = \omega^2 \wedge \omega_2^1 = (\dfrac{1}{y}dy) \wedge \omega_2^1 \therefore \omega_2^1 = -\dfrac{1}{y}dx$

$\Omega_2^1 = d\omega_2^1 = \dfrac{1}{y^2}dy \wedge dx = -\dfrac{1}{y^2}dx \wedge dy = -\omega^1 \wedge \omega^2$

所以 K=-1

2. 在R^3中使用球座標，取$\omega^r = dr, \omega^\theta = rd\theta, \omega^\phi = r\sin\theta d\phi$ 則$\omega_\phi^\theta =$

$-\cos\theta d\phi$

3. 計算S^3的 Ricci 曲率

$E_1 = \dfrac{\partial}{\partial\psi}, E_2 = \dfrac{1}{\sin\psi}\dfrac{\partial}{\partial\theta}, E_3 = \dfrac{1}{\sin\psi\sin\theta}\dfrac{\partial}{\partial\phi}$

$\omega^1 = d\psi$ ， $\omega^2 = \sin\psi d\theta$ ， $\omega^3 = \sin\psi\sin\theta d\phi$

$d\omega^1 = 0, d\omega^2 = \cos\psi d\psi \wedge d\theta, d\omega^3 = \cos\psi\sin\theta d\psi \wedge d\phi + \sin\psi\cos\theta d\theta \wedge d\phi$

$d\omega^1 = \omega^2 \wedge \omega_2^1 + \omega^3 \wedge \omega_3^1 = 0 \Rightarrow d\theta \wedge \omega_1^2 + \sin\theta d\phi \wedge \omega_1^3 = 0$

$d\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^2 = d\psi \wedge \omega_1^2 + \sin\psi \sin\theta d\phi \wedge \omega_3^2 = \cos\psi d\psi \wedge d\theta$

$\therefore \omega_1^2 = \cos\psi d\theta$

$d\omega^3 = \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = \cos\psi \sin\theta d\psi \wedge d\phi + \sin\psi \cos\theta d\theta \wedge d\phi$

$= d\psi \wedge \omega_1^3 + \sin\psi d\theta \wedge \omega_2^3$

$\therefore \omega_1^3 = \cos\psi \sin\theta d\phi$ ， $\omega_2^3 = \cos\theta d\phi$

$\Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$

$\Omega_1^2 = d\omega_1^2 - \omega_1^3 \wedge \omega_3^2 = -\omega^1 \wedge \omega^2$ $\Omega_1^3 = d\omega_1^3 - \omega_1^2 \wedge \omega_2^3 = -\omega^1 \wedge$

$\Omega_2^3 = d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = -\omega^2 \wedge \omega^3$

$R_{jkl}^i = \Omega_j^i(e_k, e_l)$ $R_{ij} = R_{ikj}^k = \sum_k \Omega_i^k(e_k, e_j)$

$R_{11} = \Omega_1^2(E_2, E_1) + \Omega_1^3(E_3, E_1) = 1 + 1 = 2$

同理 $R_{22} = R_{33} = 2$ then $R_{ij} = 2g_{ij}$

習作

1. Poincare half-plane model $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$ ，取 $\omega^1 = \frac{1}{y}dx, \omega^2 = \frac{1}{y}dy$ 則(1) $\omega_2^1 =$ (2) $\Omega_2^1 = d\omega_2^1 + \omega_k^1 \wedge \omega_2^k = ?$ (1) $-\frac{1}{y}dx$ (2) $-\omega^1 \wedge \omega^2$

2. 旋轉曲面 $ds^2 = du^2 + f(u)^2 dv^2$ ， $\omega^1 = du, \omega^2 = f(u)dv$ 若 $\omega_2^1 = -f'(u)dv$ 則該曲面的高斯曲率 K=

   $\Omega_2^1 = -K\omega^1 \wedge \omega^2 = d\omega_2^1$ $K = -\frac{f'(u)}{f(u)}$

3. 在無撓的黎曼幾何中，第一 Bianchi 恆等式以微分形是如何表達？

   $\sum_j \Omega_j^i \wedge \omega^j = 0$

4. 關於第二 Bianchi 恆等式 $d\Omega^i_j = \Omega^i_k \wedge \omega^k_j - \omega^i_k \wedge \Omega^k_j$ 其幾何意義主要描述了甚麼？

5. **在廣義相對論中，考慮靜態球對稱度量 $ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2$。選取正交余標架 $\omega^0 = e^{\Phi}dt, \omega^1 = e^{\Lambda}dr$。聯絡形式 ω^0_1 的計算結果為何？**

$\Phi' e^{\Phi-\Lambda} dt$