§ Cartan structure equation

A field of frames $\{X_1, X_2, ..., X_n\}, X_i = \frac{\partial}{\partial x^i}$

A field of dual frames $\{\omega^1, \omega^2, ..., \omega^n\}$

Levi-Civita connection $\nabla_X Y$, $\nabla_{X_i} = \sum_k \Gamma_{ij}^k X_k$

$X = \sum_i X^i \frac{\partial}{\partial x^i}, Y = \sum_i Y^i \frac{\partial}{\partial x^i}$ then $\nabla_X Y = \sum_i (XY^i + \sum_{j,k} \Gamma^i{}_{jk}) \frac{\partial}{\partial x^i}$

Riemannian curvature tensor $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Definition $\omega_j^k = \sum_i \Gamma_{ij}^k \omega^i$ or $\Gamma_{ij}^k = \omega_j^k (X_i)$ then

1. $d\omega^i = \sum_j \omega^j \wedge \omega_j^i$

2. $dg_{ij} = \sum_k (g_{kj}\omega_i^k + g_{ki}\omega_j^k)$ if $g_{ij} = \delta_{ij}$ then $\omega_i^j + \omega_j^i = 0$

3. $\Omega_i^j = d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j$

(1)(2)(3) is called Cartan structure equations

$\omega = [\omega_j^i]$ is called connection matrix。

$\Omega = [\Omega_j^i]$ is called curvature matrix。

Where $\nabla_X e_j = \sum_i \omega_j^i (X) e_i$, $R(X,Y)e_j = \sum_i \Omega_j^i (X,Y) e_i$

α, β are c^∞ 1-forms，then

$(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$

$(d\alpha)(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$

(Definition ：Let M be a smooth manifold。

Given a k-form ω in M we defined its exterior derivative $d\omega$ ：

$$d\omega(X_1, X_2, ..., X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} X_i \cdot \omega(X_1, ..., \hat{X}_i, .., X_{k+1}) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, ., \hat{X}_i, .\hat{X}_j, .., X_{k+1})$$

Where the hat indicates an omitted variable。)

Then prove $\Omega_i^j = d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j$

Let X and Y be smooth vector fields on U。Then

$$\nabla_X \nabla_Y e_j = \nabla_X \left(\sum_k \omega_j^k(Y) e_k\right)$$

$$= \sum_k X\omega_j^k(Y) e_k + \sum_k \omega_j^k(Y) \nabla_X e_k \quad (\nabla_X(fY) = (Xf)Y + f\nabla_X Y \text{ Leibniz rule})$$

$$= \sum_i X\omega_j^i(Y) e_i + \sum_{i,k} \omega^k{}_j(Y)\omega^i{}_k(X$$

Interchange X、Y gives $\nabla_Y \nabla_X e_j = \sum_i Y\omega_j^i(\mathrm{X})\mathrm{e}_i + \sum_{i,k} \omega_j^k(X)\omega_k^i e_i$

Furthermore $\nabla_{[X,Y]} e_j = \sum_i \omega_j^i([X,Y]) e_j$

Hence $R(X,Y)e_j = \nabla_X \nabla_Y e_j - \nabla_Y \nabla_X e_j - \nabla_{[X,Y]} e_j$

$$= (X\omega_j^i(Y) - Y\omega_j^i(X) - \omega_j^i([X,Y]))e_i + (\omega_k^i(X)\omega_j^k(Y) - \omega_k^i(Y)\omega_j^k(X))e_i$$

$$= d\omega_j^i(X,Y)e_i + \omega_k^i \wedge \omega_j^k(X,Y)e_i$$

$$= (d\omega_j^i + \omega_k^i \wedge \omega_j^k)(X,Y)e_i$$

Then $\Omega_i^j = d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j$

Theorem

If M is a 2-dim manifold，then for an orthonormal frame $\Omega_1^2 = -K\omega^1 \wedge \omega^2$

Where K is the Gauss curvature of M。

Cartan structure equation 的幾何意義為何，有此一說：

1. $d\omega^j = \omega^i \wedge \omega_i^j + \tau^j \Leftrightarrow \nabla_X Y - \nabla_Y X - [X,Y] = \tau(X,Y)$

2. $\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j \Leftrightarrow [\nabla_X, \nabla_Y]Z - \nabla_{[\mathrm{X,Y}]}Z = R(X,Y)Z$