Let (M,g) be a Riemannian manifold。

(a) Write down the definition of Levi-Civta connection ∇

(b) Let $\{e_i\}$ be a local frame of the tangent bundle induced by a coordinate chart。Write down the definition of Christoffel symbols Γ_{ij}^k and prove that $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$。

(a) The Levi-Civita connection on a Riemannian manifold IM,g) is the unique affine conection that is torsion-free and metric compatible。

1. Torsion-free：$\nabla_X Y - \nabla_Y X = [X,Y]$ for all vector fields X，Y
2. Metric compatible：$\nabla g = 0$ or equivalently，X(g(Y,Z))=$g(\nabla_X Y, Z)$+$g(Y, \nabla_X Z)$ for all X,Y,Z。

(b) Christoffel symbols 的歷史背景

1. 研究曲線坐標中的微分運算
2. 引入聯絡係數
3. 與高斯的曲面理論關聯
4. 與 Levi-Civita 的關係

   之後，Tullio Levi-Civita 在 1917 年進一步發展了這一概念，並確立了 Levi-Civita 聯絡，即基於度量張量導出的唯一無挫曲(torsion-free)且與度量相容的聯絡。這使得 Christoffel 符號在廣義相對論中成為關鍵工具，負責描述時空的彎曲性與自由落體運動。

證明$\Gamma_{ij}^k = \frac{1}{2} g^{lk} (\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l})$

1. 考慮度量張量在座標基底下的協變導數為零，即

   $\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma_{ik}^l g_{lj} - \Gamma_{jk}^l g_{il} = 0$ (即$\nabla g = 0$，∇ is metric compatible，或者寫成$\nabla_\lambda g_{\mu\nu} = 0$)

   將上式對索引重新排列，得到三個方程式：

   1. $\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ik}^l g_{lj} + \Gamma_{jk}^l g_{il}$
   2. $\frac{\partial g_{jk}}{\partial x^i} = \Gamma_{ji}^l g_{lk} + \Gamma_{ki}^l g_{jl}$
   3. $\frac{\partial g_{ki}}{\partial x^j} = \Gamma_{kj}^l g_{li} + \Gamma_{ij}^l g_{kl}$
2.
3. 將此三方程相加。再取g^{kl} 即可。這個推導顯示：Christoffel 符號是唯一滿足度量相容性與無挫曲條件的聯絡係數。
4. Christoffel 符號的存在確保測地線方程。

$$X = \sum_i X^i \frac{\partial}{\partial x^i}, Y = \sum_i Y^i \frac{\partial}{\partial x^i} \text{ then } \nabla_X Y = \sum_i (XY^i + \sum_{j,k} \Gamma_{jk}^i X^j Y^k) \frac{\partial}{\partial x^i}$$

Let $\{e_i\}$ be the local frame induced by a coordinate chart (U, x^i), so $e_i = \frac{\partial}{\partial x^i}$. The covariant derivative $\nabla_{e_i} e_j$ is defined by its action on the coordinate basis:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k},$$

where Γ^k_{ij} are the Christoffel symbols of the connection. Substituting $e_i = \frac{\partial}{\partial x^i}$ and $e_k = \frac{\partial}{\partial x^k}$, this becomes:

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k.$$