Martin Kruskal 與 Norman Zabusky 在 1960 年代發現孤子(soliton)在解 kdV 方程時所扮演的角色，之後___提出 KdV 方程是完全可積的(completely integrable system)解釋。 [Kolmogorov-Arnold-Moser theorem] [Peter Lax 1926-]
陀螺旋轉的運動方程式 Sonja Kovalevsky

sine-Gordon 方程 $\omega_{tt} + \sin\omega = \omega_{xx}$

KdV 方程 $u_t + uu_x + u_{xxx} = 0$

黎曼猜想 Freeman Dyson：隨機 unitary 矩陣特徵根的配對。

§ Frobenius 可積分定理

1. 微分方程的積分因子
2. 向量場的形式
3. Differential form 的形式
4. 物理學家怎麼說
5. 釋例
6. 應用

關於 Frobenius 可積分定理，先看看日新兄的說明：
Frobenius' (integrability) theorem provides an integrability condition for a system of 1-forms to vanish on an (integral) submanifold simultaneously。
That a system of 1-forms vanish simultaneously form a system of partial differential equations。
An (integral) submanifold is a solution to this system of equations。
If such a submanifold (or a solution) exists，then some integrability condition must be satisfied。
The theorem tells how to obtain such integrability condition。
This is an important and useful theorem in differential geometry。
For instance，the fundamental theorem in each known geometry (such as Riemannian or CR geometry) is proved by using this theorem (and in fact "curvature"=0 plays the role of integrability condition in this situation)。

§ 先從微分方程說起

對於一個 1-form ω，存在函數 f、g 使得 $\omega = fdg$ 的條件是什麼？
換句話說，要找 $\omega = 0$ 的積分因子。

若 $\omega = fdg$ 則 $d\omega = df \wedge dg = df \wedge f^{-1}\omega$

$d\omega = \theta \wedge \omega$，其中 $\theta = f^{-1}df = d(\ln|f|)$ 則 $d\omega \wedge \omega = \theta \wedge \omega \wedge \omega = 0$

所以$d\omega\wedge\omega=0$是$\omega=0$有積分因子的充要條件。

若$\omega=Adx+Bdy+Cdz$

$$d\omega=\begin{vmatrix} dy\wedge dz & dz\wedge dx & dx\wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & B & C \end{vmatrix}$$

$$d\omega\wedge\omega=(A(\frac{\partial C}{\partial y}-\frac{\partial B}{\partial z})+B(\frac{\partial A}{\partial z}-\frac{\partial C}{\partial x})+C(\frac{\partial \mathrm{B}}{\partial x}-\frac{\partial A}{\partial y}))dx\wedge dy\wedge dz$$

所以$Adx+Bdy+Cdz=0$有積分因子的條件是

$$A(\frac{\partial C}{\partial y}-\frac{\partial B}{\partial z})+B(\frac{\partial A}{\partial z}-\frac{\partial C}{\partial x})+C(\frac{\partial \mathrm{B}}{\partial x}-\frac{\partial A}{\partial y})=0$$

若$\omega=fdg$則微分方程$\omega=0$與$dg=0$是相同的，因此$\omega=0$的解(積分曲面)即 hypersurface g=constant。

例 1.

$\omega=dx+zdy+dz$

$d\omega=dz\wedge dy$ ，$\therefore d\omega\wedge\omega=dz\wedge dy\wedge dx=-dx\wedge dy\wedge dz\neq 0$

所以ω沒有積分因子。

例 2.

$\omega=yzdx+xzdy+z^2dz$

$$\begin{aligned} d\omega&=d(yz)\wedge dx+d(xz)\wedge dy+d(z^2)\wedge dz \\ &=(zdy+ydz)\wedge dx+(zdx+xdz)\wedge dy \\ &=ydz\wedge dx+xdz\wedge dy \end{aligned}$$

$d\omega\wedge\omega=xyzdz\wedge dx\wedge dy+xyzdz\wedge dy\wedge dx=0$

所以$\omega=0$有積分曲面。

$\omega=yzdx+xzdy+z^2dz=z(ydx+xdy+zdz)$

取 f=z，$g=xy+\frac{1}{2}z^2$則$\omega=fdg$

積分曲面是$xy+\frac{1}{2}z^2=cons\tan t$ 。

例 3.

$\omega = yzdx + xzdy + dz$

$d\omega = zdy \wedge dx + ydz \wedge dx + zdx \wedge dy + xdz \wedge dy = ydz \wedge dx + xdz \wedge dy$

$d\omega \wedge \omega = 0$

假設 $\omega = fdg = f(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz)$ 則

$$\begin{cases} f\dfrac{\partial g}{\partial x} = yz \\ f\dfrac{\partial g}{\partial y} = xz \\ f\dfrac{\partial g}{\partial z} = 1 \end{cases}$$

由(1)(2) $x\frac{\partial g}{\partial x} - y\frac{\partial g}{\partial y} = 0$，has a general solution g(z,u)，u=xy

$g = h(z)e^{xy}$，Then $dg = yh(z)e^{xy}dx + xh(z)e^{xy} + h'(z)e^{xy}dz$

$f = e^{-xy}, h(z) = z$ 所以積分曲面是 $ze^{xy} = constant$

另解

$d\omega = dz \wedge (ydx + xdy) = \frac{dz}{z} \wedge (yzdx + xzdy + dz) = (\frac{dz}{z}) \wedge \omega$

Which is not so useful since $\frac{dz}{z}$ is singular along z-axis。

A better choice is $\theta = -ydx - xdy$，then $d\omega = \theta \wedge \omega$

To determine the function g，we use the face that each integral surface g=constant will be cut by the plane {x=at, y=bt} in a curve which intersects the z-axis in the solution z of g(0,0,z)=constant。The equation $\omega = 0$ on the plane x=at, y=bt becomes dz+2abztdt=0

$E: \begin{cases} x = at \\ y = bt \end{cases}$，$dx = adt, dy = bdt$ 代入 $\omega = 0 \Rightarrow dz + 2abztdt = 0$

$\frac{dz}{dt} = -2abz$，$z = c\exp(-abt^2) = ce^{-xy}$

$dz = -yce^{-xy}dx - xce^{-xy}dy + e^{-xy}dc$

$= e^{-xy}dc - z(ydx + xdy)$

$yzdx + xzdy + dz = e^{-xy}dc = \omega = fdg$

所以 $f = e^{-xy}, g = ze^{xy}$ 積分曲面為 $ze^{xy} = constant$

We have $dz = e^{-ab}dc + c(-be^{-ab}da - ae^{-ab}db) = e^{-ab}dc - ce^{-ab}(adb + bda)$

$= e^{-ab}dc - z(adb + bda)$

得 $e^{-ab}dc = dz + z(xdy + ydx) = \omega$

$\omega = e^{-xy}d(ze^{xy})$ (note that $c = ze^{xy}$)，and the integral surfaces are $ze^{xy} =$ constant

例 4

$\omega = 2xzdx + 2yzdy + dz$

$d\omega = 2xdz \wedge dx + 2zdz \wedge dy$

$d\omega \wedge \omega = 4xyzdz \wedge dx \wedge dy + 4xyzdz \wedge dy \wedge dx = 0$

所以存在 f，g 使得 $\omega = fdg$ (f, g 不是唯一的)

解

$f\dfrac{\partial g}{\partial x} = 2xz...(1),\ f\dfrac{\partial g}{\partial y} = 2yz...(2),\ f\dfrac{\partial g}{\partial z} = 1...(3)$

由(1)(2) $\dfrac{1}{2x}\dfrac{\partial g}{\partial x} = \dfrac{1}{2y}\dfrac{\partial g}{\partial y} \Rightarrow \dfrac{\partial g}{\partial x^2} = \dfrac{\partial g}{\partial y^2}$ has a general solution g(z, u)，$u = x^2 + y^2$

由(2)(3) $\dfrac{1}{2y}\dfrac{\partial g}{\partial y} = z\dfrac{\partial g}{\partial z} \Rightarrow \dfrac{\partial g}{\partial \ln z} = \dfrac{\partial g}{\partial y^2} = \dfrac{\partial g}{\partial u}$

Hence g=G(lnz+u)，and since it is possible to pick an arbitrary function G we can set

$g = ze^{x^2+y^2}$，From (3) it follows that $f = e^{-x^2-y^2}$，and it easy to chek that

$\omega = e^{-x^2-y^2}d(ze^{x^2+y^2}) = 2xzdx + 2yzdy + dz$

積分曲面為 $ze^{x^2+y^2} =$ constant。

例 5.

$\omega = dz - ydx - dy$

On the plane x=at, y=bt，the equation $\omega = 0$ becomes dz=(abt+b)dt

$z = \dfrac{1}{2}abt^2 + bt + c$ and we arrive at the surface $z = \dfrac{1}{2}xy + y + c$

But on the parabolic cylinder x=at，y=bt^2 we have $dz = (abt^2 + 2bt)dt$

$z = \dfrac{1}{3}abt^3 + bt^2 + c$，$z = \dfrac{1}{3}xy + y + c$ a different family of surfaces。

The reason for this failure to obtain integral surfaces is seen from

$d\omega = -dy \wedge dx, d\omega \wedge \omega = -dy \wedge dx \wedge dz \neq 0$

Frobenius 可積分定理有兩種形式 (1)向量場 (2)differential forms

§1 向量場形式

§ 流形 M 上給定的一個向量場 $X \neq 0$，過 $\forall p_0 \in M$ 存在一條軌跡 $\gamma(t)$，

$$\gamma(t_0) = p_0, \frac{d\gamma}{dt} = X(\gamma(t))$$

換句話說 一個 M 上的非零向量場在 M 上每一點 p 的切空間 T_pM 上確定了一個一維子空間。
X 過 P 點的軌跡是 M 的一維子流形。

進一步推廣，在 M 上每一點 p 的切空間 T_pM 上都給一個 k 維子空間

$$L_p^k (1 \le k \le n)$$

那麼，對 M 上的每一點都存在過 P 的 k 維子流形 N$\subset M$，使得 N 上每一點 q 的切空間與給定的 k 維子空間 L_p^k 重合($T_qN = L_p^k$)的條件是甚麼 ?

Frobenius 定理回答這個問題。

定義

1. $\varphi : M \to N$ 是一個可微映射且 $d\varphi_p : T_pM \to T_{\varphi(p)}N$ 是 1-1(injective) for $\forall p \in M$ 則稱 φ 是一個浸射(immersion)。

2. 對 $\forall p \in M$，D_p 是切平面 T_pM 中的 k-dim 線性子空間。

若 $\forall p_0 \in M$，存在 $C^\infty - immersion \quad \varphi : U \to M$，

使得 $p_0 \in \varphi(U)$，且 $T_{\varphi(x)}(\varphi(U)) = D_{\varphi(x)}$ for $\forall x \in U$，則稱 D 為可積分

白話文是這麼說的：
設 $M = R^3$，在 R^3 中每一點先可微地指定一平面，得到一個 2 維平面場 D。

設通過每一點 p_0 有一個曲面 $\alpha(U)$，使得在 $\alpha(U)$ 上任一點的切平面都是原來指

定的平面，那麼 我們就稱 D 為可積的(integrable)。

3. M 是一個 n-dim 的微分流形，k 是小於 n 的正整數。D 是 M 的 k-dim 平面場(distribution 子流形)

若 $\forall X,Y \in D \Rightarrow [X,Y] \in D$ ，則稱 D 為對合的(involutive)。

另一種寫法是：

在 U 存在 local basis $X_1, X_2, \ldots X_k \in$ D 使得 $[X_i, X_j] = \sum_k c_{ij}^k X_k$

$$[X,Y] = \sum_i (XY^i - YX^i)\frac{\partial}{\partial x^i}$$

Frobenius 定理：

D 是 M 的 k 維平面場(distribution)，則 D 是可積的充要條件為 D 是對合的。

或者這麼說，L^k 是定義在 M 的一個開集 U 上的 k 維光滑分布，對任一點 $p \in U$ 存在 p 點的局部坐標系 (W, w^i) ，$W \subset U$ 使得 $L^k \Big|_W = \{\frac{\partial}{\partial w^1}, \ldots, \frac{\partial}{\partial w^k}\}$ 的充要條件是 L^k 適合 Frobenius 條件(即 L^k 對合的)。

例 $X_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial z}$

D 是 R^3 中由 X_1, X_2 所張的平面場(distribution)

$X_1 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial z}$ ，$[X_1, X_2] = 0$ ，所以 D 是對合的(inv0lutive)

X_1 的 flow 為 $\varphi_t(x,y,z) = (-y\sin t + x\cos t, y\cos t + x\sin t, z)$ 是以 z 軸為中心軸的圓。

X_2 的 flow 為 $\psi_t(x,y,z) = (x, y, t+z)$ 是平行 z 軸的直線。

所以平面場 D 的 integral manifold 是以 z 軸為中心的圓柱(cylinder)。

因為 S^2 上的連續向量場一定有奇異點(Hairy ball theorem) 所以 S^2 上沒有 1-dim 平面場。

[A Course in Modern Mathematical Physics　Peter Szekeres p.441]

Theorem 15.4 *A smooth k-dimensional distribution D^k on a manifold M is involutive if and only if every point $p \in M$ lies in a coordinate chart $(U; x^i)$ such that the coordinate vector fields $\partial/\partial x^\alpha$ for $\alpha = 1, \dots, k$ span D^k at each point of U.*

Theorem 15.5 *A set of vector fields $\{X_1, X_2, \dots, X_k\}$ is equal to the first k basis fields of a local coordinate system, $X_1 = \partial_{x^1}, \dots, X_k = \partial_{x^k}$ if and only if they commute with each other, $[X_\alpha, X_\beta] = 0$.*

(書中有證明)

例

在 R^3-{(0,0,0)}　$X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

因為 $xX_1 + yX_2 + zX_3 = 0$　此三個向量場張出一個 2-dim 平面場 D(distribution)

計算一下 $[X_1, X_2] = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = -X_3$，$[X_2, X_3] = -X_1, [X_3, X_1] = -X_2$

因此 D 是對合的，由 Frobenius 定理 存在一個 local transformation 座標

y^1, y^2, y^3，使得 D 由 $\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}$ 所張。

取 $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$，則 $[X, X_i] = 0$

因此 由 $\{X, X_1\}$ 所張的 distribution E^2 也是對合的。

D=span$\{X_1, X_2, X_3\}$ is a Lie subalgebra with [ ] of $\chi(R^3)$，isomorphic to $\{R^3, \times\}$

$F: D \to R^3$　$F(aX_1 + bX_2 + cX_3) = (a, -b, c)$

Let us consider spherical polar coordinates, Eq. (15.2), having inverse transformations

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \cos^{-1}\left(\frac{z}{r}\right), \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right).$$

Express the basis vector fields in terms of these coordinates

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\theta}{\partial x}\partial_\theta + \frac{\partial\phi}{\partial x}\partial_\phi = \sin\theta\cos\phi\,\partial_r + \frac{\cos\theta\cos\phi}{r}\partial_\theta - \frac{\sin\phi}{r\sin\theta}\partial_\phi,$$
$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial\theta}{\partial y}\partial_\theta + \frac{\partial\phi}{\partial y}\partial_\phi = \sin\theta\sin\phi\,\partial_r + \frac{\cos\theta\sin\phi}{r}\partial_\theta + \frac{\cos\phi}{r\sin\theta}\partial_\phi,$$
$$\partial_z = \frac{\partial r}{\partial z}\partial_r + \frac{\partial\theta}{\partial z}\partial_\theta + \frac{\partial\phi}{\partial z}\partial_\phi = \cos\theta\,\partial_r - \frac{\sin\theta}{r}\partial_\theta,$$

and a simple calculation gives

$$X_1 = y\partial_z - z\partial_y = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi,$$
$$X_2 = z\partial_x - x\partial_z = -\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi,$$
$$X_3 = x\partial_y - y\partial_x = \partial_\phi,$$
$$X = x\partial_x + y\partial_y + z\partial_z = r\partial_r = \partial_{r'} \quad \text{where } r' = \ln r.$$

The distribution D^2 is spanned by the basis vector fields ∂_θ and ∂_ϕ, while the distribution E^2 is spanned by the vector fields ∂_r and ∂_ϕ in spherical polars.

§2 differential forms 形式

定理

Let $\omega = \sum_i f_i dx^i$ be a one-form which does not vanish at O。

Suppose there is a one-form θ satisfying $d\omega = \theta \wedge \omega$，Then there are function f and g in a sufficiently small neighborhood of O which satisfy $\omega = fdg$

Example $\omega = xdy{-}ydx$. Certainly $\omega \wedge d\omega) = 0$ since $\omega \wedge d\omega)$ is a three-form. However, the form ω vanishes at 0 so one does not expect that the integral curves of $\omega = 0$ will span out evenly a neighborhood of 0; in fact these curves are just the lines $ax + by = 0$ through 0. We note, however, that $d\omega = \theta \wedge \omega$ is impossible in any neighborhood of 0. For $d\omega = 2\, dx\, dy$ so that if $\theta = A\, dx + B\, dy$, then $2 = Ax + By$ which fails at $x = y = 0$.

[A Course in Modern Mathematical Physics　Peter Szekeres p.455]

定理

Let $\omega^i (i = 1, 2, ..., r)$ be a set of 1-forms on an open set U，linaerly independent at every point $p \in U$。The following statements are all equivalent：

1. There exist lcal coordinates $(U; x^i)$ at every point $p \in U$ such that $\omega^i = A^i_j dx^j$

2. There exist 1-forms θ_j^i such that $d\omega^i = \sum_j \theta_j^i \omega^j$
3. $d\omega^i \wedge \Omega = 0$ where $\Omega = \omega^1 \wedge \omega^2 \wedge ... \wedge \omega^r$
4. $d\Omega \wedge \omega^i = 0$
5. There exists 1-form θ such that $d\Omega = \theta \wedge \Omega$

書中有證明

A system of linerly independent 1-forms $\omega^1,...,\omega^r$ on an open set U，satisfying any of the condition (1)-(5) of this theorem is said to be completely integrable。

The equations defining the distribution $D^k (k = n - r)$ that annihilates these ω^i is given by equations $<\omega^i, X>= 0$ ，often written as a Pfaffian system of equations $\omega^i = 0 (i = 1,...,r)$ 。

Condition (1) says that locally there exist r functions $g^i(x^1,...,x^n)$ on U such that $\omega^i = f_j^i dg^j$ 。Where the functions f_j^i form a non-singular $r \times r$ matrix at every point of U。The functions g^i are known as a first integral of the system。

The r-dimensional submanifolds (N_c, ψ_c) defined by $g^i(x^1,...,x^n) = c^i = const$ have the property $\psi_c^* \omega^i = f_j^i \circ \psi_c dc^j = 0$ and are known as integral submanifolds of the system。

Problem 16.8 Given an $r \times r$ matrix of 1-forms Ω, show that the equation

$$dA = \Omega A - A\Omega$$

is soluble for an $r \times r$ matrix of functions A only if

$$\Theta A = A\Theta$$

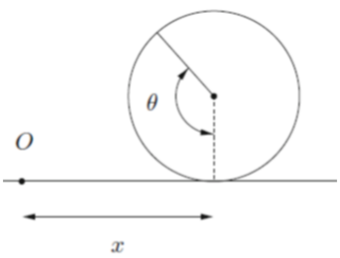
where $\Theta = d\Omega - \Omega \wedge \Omega$.

If the equation has a solution for arbitrary initial values $A = A_0$ at any point $p \in M$, show that there exists a 2-form α such that $\Theta = \alpha I$ and $d\alpha = 0$.

§ 在分析力學中

$\omega=0$是線性速度約束，若$d\omega\wedge\omega=0$則該動力系統的速度約束為可積，該系統為具有完整約束的動力系統。
約束 (constraint)，可積的(holonomic)

[An introduction to Riemannian Geometry p.198] 幾何力學
Theorem 4.8
A ditribution Σ is integrable if and only if $X,Y\in\chi(\Sigma)\Rightarrow[X,Y]\in\chi(\Sigma)$
例

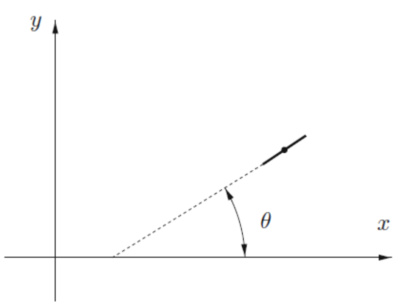


Wheel rolling without slipping
(slipping 打滑；slipper 拖鞋)
Consider a vertical wheel of radius R rolling without slipping on a plane。Assuming that the motion takes place along a straight line，we can parameterize any position of the wheel by the position x of contact point and the angle θ between a fixed radius of the wheel and the radius containing the contact point，hence the configuratin space is $\mathrm{R}\times\mathrm{S}^1$。

Then $\dot{x}=R\dot{\theta}$，this is equvalent to requiring that the motion be compatible with the distribution defined on $R\times S^1$ by the vector field $X=R\frac{\partial}{\partial x}+\frac{\partial}{\partial\theta}$ or equivalently，by the kernel of the 1-form $\omega=dx-Rd\theta$。
Since $d\omega=0$，we see that is a semi-holonomic constraint，corresponding to an integrable distribution。

The leaves of the distribution are the submanifolds with equatuin $x=x_0+R\theta$

例 溜冰(ice skate)



一個溜冰的簡單模型是沿本身(冰刀)移動或以中心旋轉，冰刀的位置可以用冰刀中心點座標(x,y)與冰刀與 x 軸的夾角θ來表示。如上圖。因此 其組態空間(configuration space)為$R^2\times S$

若冰刀只沿本身移動，則$(\dot{x},\dot{y})$與$(\cos\theta,\sin\theta)$成比例，$R^2\times S$的平面場

(distribution) $\Sigma=\{X,Y\}$ ，其中 $X=\cos\theta\frac{\partial}{\partial x}+\sin\theta\frac{\partial}{\partial y}, Y=\frac{\partial}{\partial\theta}$

或者 $\Sigma=\ker(\omega)$ ， $\omega=-\sin\theta dx+\cos\theta dy$

$d\omega\wedge\omega=-\cos^2\theta d\theta\wedge dx\wedge dy+\sin^2\theta d\theta\wedge dy\wedge dx$

$=-d\theta\wedge dy\wedge dx\neq 0$

由 one-form ω 的 kernel 給定的約束是不可積約束(non-holonomic constraint)。

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習作

Let $\alpha=dz+xdy-ydx\in\Omega^1(R^3)$ 。

Consider the distribution $E\subset TM$ defined by

$$E_p=\left\{v_p\in T_p\mathrm{R}^3\,\middle|\,\alpha_p(v_p)=0\right\}\text{ ， }p\in R^3$$

Determine whether or not E is integrable。Prove your answer。

$d\alpha=-dx\wedge dy-dy\wedge dx=0$

設 $\alpha=fdg=f(\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy+\frac{\partial g}{\partial z}dz)$ ，then

$$\begin{cases} f\frac{\partial g}{\partial x}=-y \\ f\frac{\partial g}{\partial y}=-x\Rightarrow x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}=0\Rightarrow g=h(z)e^{\frac{y}{x}} \\ f\frac{\partial g}{\partial z}=1 \end{cases}$$

$$dg = h(z)e^{\frac{y}{x}} \times (\frac{-y}{x^2})dx + h(z)e^{\frac{y}{x}} \times \frac{1}{x}dy + h'(z)e^{\frac{y}{x}}dz$$

取 $f = \frac{x^2}{z}e^{-\frac{y}{x}}, h(z) = z$ ，所以 E 是可積的，其 integral surface($\alpha = 0$ 的解) 是

hypersurface $ze^{\frac{y}{x}} = constant$