微分方程的可積性

用微分形式表達的 Frobenius 定理。
用 differential form 來討論積分子流形有其方便性。

M 是 N 的嵌入子流形，切空間$T_p(M)$被推前，餘切空間$T_p^*(N)$被拉回。

Notice that dim(N)>dim(M)，some cotangent vectors must be zero when they restricted on the submanifold M。
餘切場的可積問題即 N 上的 Pfaff 方程$\omega=\omega^i dx^i=0...(1)$ 的解的存在問題，此處$\omega\in\Lambda^1(N)$
意思是$\varphi:M\to N$ 使得$\varphi^*\omega$=0
找積分子流形存在的條件即找(1)可解的條件。

1. 在 P 點鄰域 U 可選坐標系，使存在函數$\varphi(x)$滿足$\omega=d\varphi$
   此時φ=constant 是$\omega=0$的積分流形，是流形上的超曲面(hypersurface)。
2. 在 P 點鄰域 U 可選坐標系，使存在函數 u(x)，v(x)滿足$\omega=udv$
   此時u^{-1}即積分因子。v=constant 是$\omega=0$的積分曲面。

Poincare theorem：

定理 2.3(Poincare) r **形式(**$r\geqslant1$**)**α **如滿足** $d\alpha=0$**,則在一点邻域存在(**$r-1$**)形式** β **使**$\alpha=d\beta$.

闭形式局域为正合形式,而整体不一定,整体拓扑性质是将在第七章分析的问题.

Frobenius theorem：
$\omega\wedge d\omega=0$是方程$\omega=0$可積的必要條件。

1. 例如在R^2上，$\omega=P(x,y)dx+Q(x,y)dy$

2. R^3上的 1-form $\omega=P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$

$$d\omega=\begin{vmatrix} dy\wedge dz & dz\wedge dx & dx\wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$\omega=0$有解(有積分因子)的條件是$p(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})+Q(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})+R(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})=0$
當$\omega\wedge d\omega=0$則ω可表為$\omega=\psi(x,y,z)d\varphi(x,y,z)$
$\varphi(x,y,z)$=constant 是R^3中的 2 維閉子流形。
定理

若可微分布(distriution)D^k為對合分布，則其對偶分布$\tau_{n-k}=\{\theta^a\}_1^{n-k}$ 必滿足以下

三個條件之一：

1. 在點 P 鄰域存在 $(n-k)^2$ 個 1-form $\alpha_b^a \in \Lambda^1(N)$ 滿足 $d\theta^a = \sum_{b=1}^{n-k} \alpha_b^a \wedge \theta^b$，
   a=1,2,⋯,n-k
2. 餘分布 τ_{n-k} 的基形式 $\{\theta^a\}_1^{n-k}$ 線性獨立，故 $\Sigma \equiv \theta^1 \wedge ... \wedge \theta^{n-k} \neq 0$

   而其每個元素 $\theta^a \in \tau_{n-k}$ 滿足 $d\theta^a \wedge \Sigma = 0$
3. 在點 P 鄰域存在光滑函數 $g_b^a(x)$， $f^a(x) \in F(N)$ 使得 $\theta^a = \sum_{b=1}^{n-k} g_b^a df^b$，
   a=1,2,⋯,n-k

以上三種條件為等價，都是餘分布 $\tau_{n-k} = \{\theta^a\}_1^{n-k}$ 的可積條件。

定理 2.5 （Frobenius 定理）Pfaff 方程组：

$$\theta^a = 0, \quad a = 1, \cdots, n-k$$

完全可积的充要条件是，在普点 p 邻域，由余分布 $\{\theta\}_1^{n-k}$ 生成的理想是闭理想. 即 $d\theta^a$ 满足下两条件之一：
1）存在 1 形式 $\{\sigma_b^a\}$，使

$$d\theta^a = \sum_{b=1}^{n-k} \sigma_b^a \wedge \theta^b \qquad (2.75)$$

或 2)令 $\Sigma = \theta^1 \wedge \cdots \wedge \theta^{n-k}$，而

$$d\theta^a \wedge \Sigma = 0 \qquad (2.76)$$

这是用微分形式表达的 Frobenius 定理.

後面有 PDE 可積條件的例子。P.68
以 KdV 方程為例：$u_t + 6uu_x + u_{xxx} = 0$

1. The Frobenius theorem states that a system of partial differential equations is integrable if and only if the associated distribution (a collection of vector fields) is **involutive**，meaning that the Lie bracket of any two vector fields in the distribution remains within the distribution。

   In simpler terms，the system is integrable if the vector fields defining the system "close" under commutation。

   For a system of PDEs，this translates to the existence of a**foliation**(a decomposition into submanifolds) such that the solutions to the system lie on these submanifolds。

2. The KdV equation is given by：$ut+uxxx+6uux=0$,where $u(x,t)$ is a function of spacex$x$and timet t。 The KdV equation is known to be**integrable**，meaning it

possesses an infinite number of conserved quantities and can be solved exactly using techniques like the**inverse scattering transform**。

To connect this to the Frobenius theorem，we need to interpret the KdV equation as a dynamical system on an infinite-dimensional manifold (the space of functions $u(x)$) and analyze its associated vector fields。

3. The KdV equation can be viewed as an evolution equation on an infinite-dimensional manifoldM, where each point onMrepresents a function $u(x)$！ The time evolution of $u(x,t)$ is governed by a vector field X onM，defined by: $X(u)=-u_{xxx}-6uu_x$。

   The integrability of the KdV equation can be understood in terms of the existence of a**hierarchy of commuting vector fields**onM。These vector fields correspond to the infinite sequence of conserved quantities (Hamiltonians) associated with the KdV equation。

4. The Frobenius theorem can be applied to the KdV equation by considering the**involutivity**of the vector fields associated with its hierarchy of conserved quantities。

   Each conserved quantity generates a flow (a vector field), and the KdV equation is integrable because these flows**commute**with each other。

   This means that the Lie bracket of any two vector fields in the hierarchy vanishes: $[X_i, X_j]=0$，where X_i and X_j are vector fields corresponding to different conserved quantities。This commutativity ensures that the system is integrable in the sense of Frobenius。

5. Using the Frobenius theorem，the integrability of the KdV equation can be interpreted as the existence of an infinite-dimensional foliation of the manifoldMby submanifolds (level sets of the conserved quantities)，such that the vector fields associated with the KdV hierarchy are tangent to these submanifolds and commute with each other。This geometric perspective provides a deep understanding of the structure and solvability of the KdV equation。