§ Stokes 定理 $\int_D \partial w = \int_{\partial D} w$

梯度 旋度 散度

Poincare lemma $d(dw)=0$

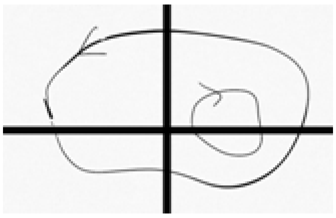
一.

f 是可微分函數 0-form

w=f,D=[a,b],$dw = df = f'dx$

則 $\int_D dw = \int_D f'(x)dx = \int_{\partial D} f = f(b)-f(a)$ 這是微積分基本定理

二.



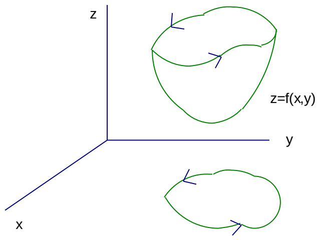
w=Pdx+Qdy 1-form

$$dw = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy$$

則 $\int_{\partial D} Pdx + Qdy = \int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy$ ...Green 定理

三.

D 是 R^3 中的有向曲面,∂D是有向閉曲線



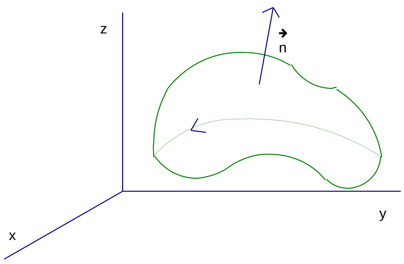
$w = Pdx + Qdy + Rdz$ ...1-form

$$dw = \begin{vmatrix} dy \wedge dz & dz \wedge dx & dx \wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$,則

$\oint \overline{E} \cdot \vec{t} ds = \iint_S (curl \overline{E}) \cdot \vec{n} dS$ ...Stokes 定理

四.

D 是 R^3 中的有界區域



$w = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ ...2-form

則 $dw = (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dx \wedge dy \wedge dz$

$\iint_S \overline{E} \cdot \vec{n} dS = \iiint_V div \overline{E} dV$ ...Gauss 定理(散度定理)

[誰怕向量微積分 Div,Grad,Curl,and All That H.M.Schey]

在靜電場中

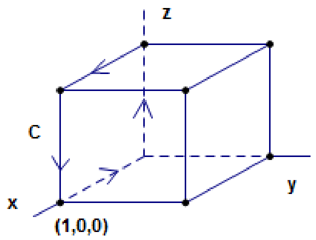
高斯定律 $\iint_S \mathrm{E} \cdot \vec{n} dS = 4\pi q$, $\iint_S F \cdot \vec{n} dS$ 稱為通量(Flux)

$$divE = \lim_{\substack{\Delta V \to 0 \\ about(x,y,z)}} \frac{1}{\Delta V} \iint_S E \cdot \vec{n} dS = 4\pi\rho \text{(單位體積的通量)} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

可以推出連續方程 $\frac{\partial \rho}{\partial t}+\nabla \cdot J=0$，其中 $J=\rho V$

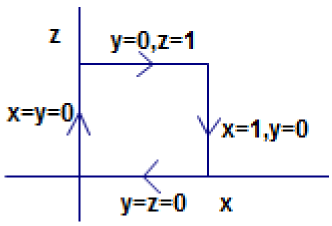
例1． Stokes 定理習作

$F=[z^2,-y^2,0]$



驗證 $\oint_C F\cdot t ds=\iint_S \vec{n}\cdot curlF dS$

$$curlF=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & -y^2 & 0 \end{vmatrix}=[0,2z,0]$$



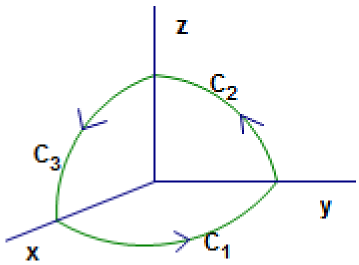
$$\oint_C \mathrm{F}\cdot t ds=\oint (F_x dx+F_y dy+F_z dz)$$

$$=\oint_C (z^2 dx-y^2 dy)=\int_0^1 dx=1$$

覆蓋面 S1~S5 只有 S5，$\vec{n}=[0,1,0]$，積分值不為 0

$$\iint_{S_5} \vec{n}\cdot curlF dS=\int_0^1\int_0^1 2z dx dz=1$$

例2．



F=[y,z,x]

S：$x^2+y^2+z^2=1$

$0\le x,y,z\le 1$

C 由 C_1,C_2,C_3 構成

$$\oint_C F\cdot t ds=\oint_C (y dx+z dy+x dz)$$

$$curlF=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}=[-1,-1,-1]$$ ，S：$x^2+y^2+z^2=1$，$\vec{n}=[x,y,z]$

$$z=f(x,y)=\sqrt{1-x^2-y^2}\ ,\ \frac{\partial f}{\partial x}=-\frac{x}{z},\frac{\partial f}{\partial y}=-\frac{y}{z}$$

$$\iint_S \vec{n}\cdot curlF dS=\iint_R -(x+y+\sqrt{1-x^2-y^2})\sqrt{1+(\frac{\partial f}{\partial x})^2+(\frac{\partial f}{\partial y})^2}dxdy$$

Let $x = r\cos\theta, y = r\sin\theta, 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\int_0^{\frac{\pi}{2}}\int_0^1 -(r\cos\theta + r\sin\theta + \sqrt{1-r^2})\frac{1}{\sqrt{1-r^2}} r dr d\theta$$

$$\oint_C (ydx + zdy + xdz) = \frac{3\pi}{4}$$

$$\int \frac{x^2 dx}{\sqrt{1-x^2}} = -\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x + C$$

$$\int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr = -\frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\sin^{-1} r \Big|_0^1 = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \frac{\pi}{4}(\cos\theta + \sin\theta) d\theta = \frac{\pi}{4}(\sin\theta - \cos\theta)\Big|_0^{\frac{\pi}{2}} = \frac{2\pi}{4}$$

$$\int_0^{\frac{\pi}{2}}\int_0^1 r dr d\theta = \frac{\pi}{4}$$

例3． 求 $\iint_S z^2 dS$ ，其中 S 是在第一卦限的 $\frac{1}{8}$ 球面 $\frac{\pi}{6}$

$$X(x,y) = (x, y, \sqrt{1-x^2-y^2})$$

$$X_x = [1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}]$$

$$X_y = [0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}]$$

$E = \frac{1-y^2}{1-x^2-y^2}$, $F = \frac{xy}{1-x^2-y^2}$, $G = \frac{1-x^2}{1-x^2-y^2}$

$$dS = \sqrt{EG - F^2} dxdy = \frac{1}{z} dxdy$$

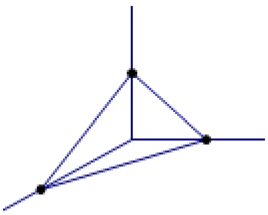
$$x = r\cos\theta, y = r\sin\theta$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\iint_S z^2 dS = \iint_R \sqrt{1-x^2-y^2}\,dxdy = \int_0^{\frac{\pi}{2}}\int_0^1 r\sqrt{1-r^2}\,drd\theta = \frac{1}{3}\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{6}$$

例4． 求 $\iint_S F\cdot\vec{n}dS$ ，其中 S 是 x+2y+2z=1 與第一卦限截出之曲面

F=(z,-y,x) $\frac{1}{2}$



$$X = [x, y, 1-\frac{1}{2}x-y]$$

$$X_x = [1, 0, -\frac{1}{2}], X_y = [0, 1, -1]$$

$$E = \frac{5}{4}, F = \frac{1}{2}, G = 2 \text{ , } dS = \sqrt{EG-F^2}\,dxdy = \frac{3}{2}dxdy$$

$$\vec{n} = [\frac{1}{3}, \frac{2}{3}, \frac{2}{3}] \text{ , } F\cdot\vec{n} = \frac{1}{3}z - \frac{2}{3}y + \frac{2}{3}x = \frac{1}{3} + \frac{1}{2}x - y$$

$$\iint_S F\cdot\vec{n}dS = \iint_R \frac{3}{2}(\frac{1}{3}+\frac{1}{2}x-y)dxdy$$

$$\int_0^{2-2y}(\frac{1}{2}+\frac{3}{4}x-\frac{3}{2}y)dx = \ldots = \frac{9}{2}y^2 - 7y + \frac{5}{2}$$

$$\iint_R \frac{3}{2}(\frac{1}{3}+\frac{1}{2}x-y)dxdy = \int_0^1(\frac{9}{2}y^2-7y+\frac{5}{2})dy = \frac{1}{2}$$

在 R^3 中 $\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}]$

$$gradf = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}] \qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$curlF = \nabla\times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$\omega = Pdx + Qdy + Rdz$ 則 $d\omega = \begin{vmatrix} dy\wedge dz & dz\wedge dx & dx\wedge dy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

$divF = \nabla\cdot F = \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z}$

$\omega = Pdy\wedge dz + Qdz\wedge dx + Rdx\wedge dy$ 則 $d\omega = (\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z})dx\wedge dy\wedge dz$

則 curl(gradf)=0,div(curlF)=0,即 R^3 中 $d(d\omega)=0$

Poincare lemma

M 是可縮的

(1) $\int_I \omega$ 與路徑無關 (2) $d\omega = 0$ (3) $\exists\varphi$ 使得 $\omega = d\varphi$ 等價

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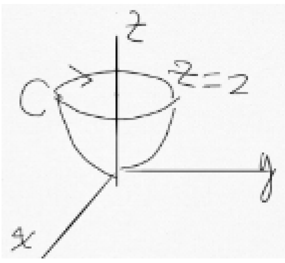
在電路 C 上的電動勢 ε 相當於電場 E 在此電路上的環積 $\varepsilon = \oint_C E\cdot tds$

法拉第發現在一穩定電路上 $\varepsilon = -\dfrac{1}{c}\dfrac{d\phi}{dt}, \phi = \iint_S \mathrm{B}\cdot\vec{n}dS$,試導出 $\nabla\times E = -\dfrac{1}{c}\dfrac{\partial B}{\partial t}$

$\varepsilon = \oint_C \mathrm{E}\cdot tds \overset{Stokes}{=} \iint_S (\nabla\times\mathrm{E})\cdot\vec{n}dS$

另一方面 $\varepsilon = -\dfrac{1}{c}\dfrac{d\phi}{dt} = -\dfrac{1}{c}\dfrac{d}{dt}\iint_S B\cdot\vec{n}dS = \iint_S (-\dfrac{1}{c}\dfrac{\partial B}{\partial t})\cdot\vec{n}dS$

剝掉 $\iint$ 得 $\nabla\times E = -\dfrac{1}{c}\dfrac{\partial B}{\partial t}$



A=[3y,-xz,yz^2]

S 2z=x^2+y^2

C: $\begin{cases} x^2 + y^2 = 4 \\ z = 2 \end{cases}$

Check Stokes 定理

$\oint_C A\cdot tds = \iint_S (\nabla\times A)\cdot\vec{n}dS$

左式是 $\int_{\partial D} w$, 其中 $w = 3ydx - xzdy + yz^2dz$

$$\begin{cases} x = 2\cos t \\ y = 2\sin t, 0 \le t \le 2\pi \\ \quad z = 2 \end{cases}$$,則 $w = ... = (-10 + 2\cos 2t)dt$

C 是順時鐘方向,所以是 $\int_{2\pi}^{0} (-10 + 2\cos 2t)dt = 20\pi$

右式 $curlA = \nabla \times A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix} = [z^2 + x, 0, -z-3]$

$z = f(x, y) = \frac{1}{2}(x^2 + y^2)$

$dS = \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}\, dx \wedge dy = \sqrt{1 + x^2 + y^2}\, dx \wedge dy$

$\vec{n} = \frac{\nabla(x^2 + y^2 - 2z)}{\left|\nabla(x^2 + y^2 - 2z)\right|} = \frac{1}{\sqrt{x^2 + y^2 + 1}}[x, y, -1]$

所以 $\iint_S (\nabla \times A) \cdot \vec{n} dS = \iint_R (xz^2 + x^2 + z + 3) dxdy$, let $x = r\cos\theta, y = r\sin\theta$

$= \int_0^{2\pi} \int_0^2 (\frac{1}{4} r^5 \cos\theta + r^2 \cos^2\theta + \frac{r^2}{2} + 3) r dr \wedge d\theta$

$= 20\pi$

(注意 $dx \wedge dy = \frac{\partial(x, y)}{\partial(r, \theta)} dr \wedge d\theta = r dr \wedge d\theta$,其中 $\frac{\partial(x, y)}{\partial(r, \theta)} = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} = r$)

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D: $\begin{cases} x = x(u, v) \\ y = y(u, v) \\ = z(u, v) \end{cases}$, $\omega = Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx$

$\int_D \omega = \iint_D Pdx \wedge dy + Qdy \wedge dz + Rdz \wedge dx$

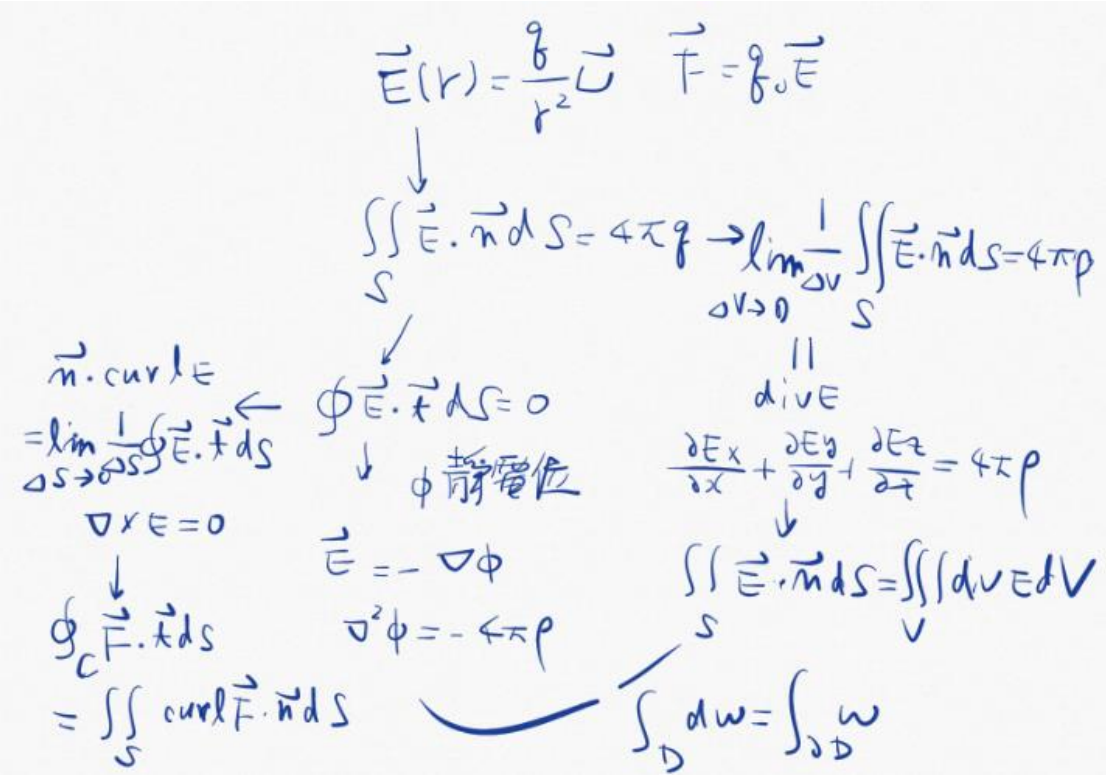
$= \iint_{D'} (P\frac{\partial(x, y)}{\partial(u, v)} + Q\frac{\partial(y, z)}{\partial(u, v)} + R\frac{\partial(z, x)}{\partial(u, v)}) du \wedge dv$

其中 $\frac{\partial(x,y)}{\partial(u,v)}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$，以此類推

$$d\omega=\begin{vmatrix} dy\wedge dz & dz\wedge dx & dx\wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

前頁就是在 check $\int_{\partial D}\omega=\int_D d\omega$

§ 靜電場



§

Let x、y、z be the restrictions of the Cartesian coordinate functions in R^3 to S^2，oriented so that {(1,0,0)；(0 ,1,0)is a positively oriented basis of $T_{(0,0,1)}S^2$，and consider the 2-form $\omega=xdy\wedge dz+ydz\wedge dx+zdx\wedge dy\in\Omega^2(S^2)$

Compute the integral $\int_{S^2}\omega=$

$$\psi:[0,2\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}]\underset{\psi^*}{\overset{\psi}{\rightleftarrows}}S^2$$

$\psi(\theta,\varphi)=(\cos\theta\cos\varphi,\sin\theta\cos\varphi,\sin\varphi)$，Then

$$\psi^*dx = \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\varphi}d\varphi = -\sin\theta\cos\varphi d\theta - \cos\theta\sin\varphi d\varphi$$

$$\psi^*dy = \frac{\partial y}{\partial\theta}d\theta + \frac{\partial y}{\partial\varphi}d\varphi = \cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi$$

$$\psi^*dz = \frac{\partial z}{\partial\theta}d\theta + \frac{\partial z}{\partial\varphi}d\varphi = \cos\varphi d\varphi$$

$$\begin{aligned}\psi^*\omega &= x(\psi^*dy\wedge\psi^*dz) + y(\psi^*dz\wedge\psi^*dx) + z(\psi^*dx\wedge\psi^*dy)\\ &= (\cos\theta\cos\varphi)(\cos\theta\cos^2\varphi)d\theta\wedge d\varphi\\ &\quad + (\sin\theta\cos\varphi)(\sin\theta\cos^2\varphi)d\theta\wedge d\varphi\\ &\quad + (\sin\varphi)[\sin^2\theta\sin\varphi\cos\varphi + \cos^2\theta\sin\varphi\cos\varphi)d\theta\wedge d\varphi\\ &= \cos\varphi d\theta\wedge d\varphi\end{aligned}$$

$$\int_{S^2}\omega = \int_U \psi^*\omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\cos\varphi d\theta d\varphi = 4\pi$$

$$d\omega = 3dx\wedge dy\wedge dz$$

$\int_{S^2}\omega = \int_\Omega d\omega = 3\int_\Omega dx\wedge dy\wedge dz = 4\pi$ 是單位球的 3 倍體積。

這驗證了 Stokes 定理 $\int_{\partial\Omega}\omega = \int_\Omega d\omega$ ，其中 $S^2 = \partial\Omega$ ，Ω 是實心球。