Differential Forms

§1 外微分 $d\omega$

$\omega = a_I dx^I$ is a k-form，then $d\omega = \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$

把 d 當作一個算子，$d = \frac{\partial}{\partial x^\mu} dx^\mu$

一. $f : R^n \to R$ is a 0-form then

$$df = \frac{\partial f}{\partial x^j} dx^j = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$ ...梯度(grad) is a 1-form

二. $\omega = a_i dx^i$ is a 1-form，then

$$d\omega = (\frac{\partial a_i}{\partial x^j} dx^j) \wedge dx^i$$ is a 2-form

$A = A_\mu dx^\mu$ is a 1-form，then

$$dA = dx^\mu \frac{\partial A}{\partial x^\mu} = dx^\mu \frac{\partial}{\partial x^\mu}(A_\nu dx^\nu) = \frac{\partial A_\nu}{\partial x^\mu} dx^\mu dx^\nu$$

When n=3，$d\omega = \begin{vmatrix} dx^2 \wedge dx^3 & dx^3 \wedge dx^1 & dx^1 \wedge dx^2 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ a_1 & a_2 & a_3 \end{vmatrix}$ ...旋度(curl)

三. $\omega = a_{ij} dx^i \wedge dx^j$ is a 2-form，then $d\omega = (\frac{\partial a_{ij}}{\partial x^k} dx^k) \wedge dx^i \wedge dx^j$

When n=3，$d\omega = (\frac{\partial a_{23}}{\partial x^1} + \frac{\partial a_{31}}{\partial x^2} + \frac{\partial a_{12}}{\partial x^3}) dx^1 \wedge dx^2 \wedge dx^3$ ...散度(div)

例1. $\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$，in $R^2/\{O\}$，求 $d\omega$

It is easy to check $d\omega = 0$

If exists a 0-form φ such that $\omega = d\varphi$

Note that $\int \frac{dx}{1+x^2} = \arctan x$，then $\varphi = \arctan(\frac{y}{x})$

$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = \omega$，$\varphi_x = \frac{-y}{x^2+y^2}, \varphi_y = \frac{x}{x^2+y^2}$

Note that $(\tan^{-1}x)'=\dfrac{1}{1+x^2}$ ， $\dfrac{d}{dx}(\tan^{-1}\dfrac{y}{x})=\dfrac{1}{1+(\dfrac{y}{x})^2}\times(\dfrac{-y}{x^2})=\dfrac{-y}{x^2+y^2}$

$\varphi=\tan^{-1}\dfrac{y}{x}+h(y)$ ， $\varphi_y=\dfrac{x}{x^2+y^2}+h'(y)$ ， $\therefore h'(y)=0$

$\varphi(x,y)=\tan^{-1}\dfrac{y}{x}+c$ isn't defined at y-axis/{O}

So ω is closed，but not exact in $R^2/\{O\}$

Poincare lemma：
M is a contractible smooth manifold then every closed form is exact。

習作

1. Check $(Adx+Bdy+Cdz)\wedge(Pdx+Qdy+Rdz)=\begin{vmatrix} dy\wedge dz & dz\wedge dx & dx\wedge dy \\ A & B & C \\ P & Q & R \end{vmatrix}$

2. ω是 1-form，證明 $d\omega(X\wedge Y)=X\omega(Y)-Y\omega(X)-\omega([X,Y])$

前面有一個習作
若$\eta=p(dy\wedge dz)+q(dz\wedge dx)+r(dx\wedge dy)$
$X=u^1\dfrac{\partial}{\partial x}+u^2\dfrac{\partial}{\partial y}+u^3\dfrac{\partial}{\partial z}$ ， $Y=v^1\dfrac{\partial}{\partial x}+v^2\dfrac{\partial}{\partial y}+v^3\dfrac{\partial}{\partial z}$ 則

$\eta\cdot(X\wedge Y)=\begin{vmatrix} p & q & r \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix}$ …(*)

取$\omega=hdx$ 驗證即可
則$d\omega=(\dfrac{\partial h}{\partial x}dx+\dfrac{\partial h}{\partial y}dy+\dfrac{\partial h}{\partial z}dz)\wedge dx=\dfrac{\partial h}{\partial z}dz\wedge dx-\dfrac{\partial h}{\partial y}dx\wedge dy$

$\omega\cdot X=hdx(u^1\dfrac{\partial}{\partial x}+u^2\dfrac{\partial}{\partial y}+u^3\dfrac{\partial}{\partial z})=hu^1$ ，同理$\omega\cdot Y=hv^1$

Then $X(\omega\cdot Y)=X(hv^1)=u^1(v^1\dfrac{\partial h}{\partial x}+h\dfrac{\partial v^1}{\partial x})+u^2(v^1\dfrac{\partial h}{\partial y}+h\dfrac{\partial v^1}{\partial y})+u^3(v^1\dfrac{\partial h}{\partial z}+h\dfrac{\partial v^1}{\partial z})$

同理 $Y(\omega\cdot X)=v^1(u^1\dfrac{\partial h}{\partial x}+h\dfrac{\partial u^1}{\partial x})+v^2(u^1\dfrac{\partial h}{\partial y}+h\dfrac{\partial u^1}{\partial y})+v^3(u^1\dfrac{\partial h}{\partial z}+h\dfrac{\partial u^1}{\partial z})$

$\omega\cdot[X,Y]=h(Xv^1-Yu^1)$

$$=h(u^1\frac{\partial v^1}{\partial x}+u^2\frac{\partial v^1}{\partial y}+u^3\frac{\partial v^1}{\partial z}-v^1\frac{\partial u^1}{\partial x}-v^2\frac{\partial u^1}{\partial y}-v^3\frac{\partial u^1}{\partial z})$$

(1)－(2)－(3)得

$$X(\omega\cdot Y)-Y(\omega\cdot X)-\omega\cdot[X,Y]=(u^2v^1-u^1v^2)\frac{\partial h}{\partial y}+(u^3v^1-u^1v^3)\frac{\partial h}{\partial z}$$

$$=\begin{vmatrix} 0 & \frac{\partial h}{\partial z} & -\frac{\partial h}{\partial y} \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix}=d\omega\cdot(X\wedge Y)$$

$\omega=\sum_i h_i dx^i$ ， $X=\sum_i X^i\frac{\partial}{\partial x^i},Y=\sum_i Y^i\frac{\partial}{\partial x^i}$

則 $\omega(X)=\sum_i h_i X^i,\omega(Y)=\sum_i h_i Y^i$

$[X,Y]=\sum_i(XY^i-YX^i)\frac{\partial}{\partial x^i}$ ， $X\wedge Y=\sum_{i,j}(X^iY^j-X^jY^i)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$

取 $\omega=hdx^k$ ，則

$\omega(X)=hX^k,\omega(Y)=hY^k$ ， $d\omega=dh\wedge dx^k=\sum_i\frac{\partial h}{\partial x^i}dx^i\wedge dx^k$

$$X\omega(Y)=\sum_i X^i\frac{\partial}{\partial x^i}(hY^k)=\sum_i X^i(h\frac{\partial Y^k}{\partial x^i}+Y^k\frac{\partial h}{\partial x^i})$$

$$Y\omega(X)=\sum_i Y^i\frac{\partial}{\partial x^i}(hX^k)=\sum_i Y^i(h\frac{\partial X^k}{\partial x^i}+X^k\frac{\partial h}{\partial x^i})$$

$$X\omega(Y)-Y\omega(X)=h\sum_i(X^i\frac{\partial Y^k}{\partial x^i}-Y^i\frac{\partial X^k}{\partial x^i})+\sum_i(X^iY^k-X^kY^i)\frac{\partial h}{\partial x^i}$$

$$=\omega[X,Y]+d\omega(X\wedge Y)$$

舉例驗證

$X=y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}$ ， $Y=z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}$ ，$\omega=ydx-xdz$

$$[X,Y]=(XY^i-YX^i)\frac{\partial}{\partial^i}=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}$$

As an exercise：

If $\eta=p(dy\wedge dz)+q(dz\wedge dx)+r(dx\wedge dy)$

$X=u^1\frac{\partial}{\partial x}+u^2\frac{\partial}{\partial y}+u^3\frac{\partial}{\partial z}$ ， $Y=v^1\frac{\partial}{\partial x}+v^2\frac{\partial}{\partial y}+v^3\frac{\partial}{\partial z}$

Then $\eta\cdot(X\wedge Y)=\begin{vmatrix} p & q & r \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix}$

Now check the formula

$d\omega = dy\wedge dx - dx\wedge dz$

$$d\omega(X\wedge Y) = \begin{vmatrix} 0 & 1 & -1 \\ 0 & -z & y \\ z & 0 & -x \end{vmatrix} = z(y-z)$$

$$\omega(X) = (ydx - xdz)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = -xy$$

$$\omega(Y) = (ydx - xdz)\left(z\frac{\partial}{x} - x\frac{\partial}{\partial z}\right) = yz + x^2$$

$$\omega([X,Y]) = (ydx - xdz)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) = y^2$$

$$X\omega(Y) = \left(y\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}\right)(yz + x^2) = y^2 - z^2$$

$$Y\omega(X) = \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)(-xy) = -yz$$

$X\omega(Y) - Y\omega(X) - \omega([X,Y]) = z(y-z)$

It has been checked。

$$X\wedge Y = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ 0 & -z & y \\ z & 0 & -x \end{vmatrix}$$

$d\omega(X\wedge Y) = (dy\wedge dx - dx\wedge dz)(X\wedge Y) = (-dx\wedge dy) + dz\wedge dx)\big(xz\partial_x + yz\partial_y + z^2\partial_z\big) = -z^2 + yz$

Note that $(dx\wedge dy)\partial_z = 1, (dx\wedge dy)\partial_x = 0$

§2 Lie derivative of differential form ω along X

$$L_X\omega = \frac{d}{dt}(\varphi_t^*\omega)_{t=0} = \sum_j (Xh_j)\, dx^j + \sum_k h_k dX^k$$

1. $d(\omega+\eta) = d\omega + d\eta$
2. $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^{\deg\omega}(\omega\wedge d\eta)$
3. $d(d\omega) = 0$
4. $\varphi: R^m \to R^n$，then $d(\varphi^*\omega) = \varphi^*(d\omega)$
5. $L_X(d\omega) = d(L_X\omega)$

6. $L_X\omega(Y) = L_X\omega(Y) - \omega(L_X Y)$

How about the computation of $L_X g$ ，where g is the metric。For a Killing vector field X， $L_X g = 0$

§3 pull-back a form

Example

$x = r\cos\theta, y = r\sin\theta$

$dx = \cos\theta dr - r\sin\theta d\theta$

$dy = \sin\theta dr + r\cos\theta d\theta$

Then $dx \wedge dy = rdr \wedge d\theta$

$ds^2 = dr^2 + r^2 d\theta^2$ ， $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$dA = \sqrt{\det g_{ij}}\, dr \wedge d\theta = rdr \wedge d\theta$

$R^2 \xrightarrow{\varphi} R^2 \quad (x, y) \to (r, \theta)$

$d\,x \wedge \;\; d \quad d\,A = \;\; r\,d\wedge r \;\; \theta$

$\varphi^* dA = dx \wedge dy$

§4 Spherical coordinates

$x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$

Then $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2$

$$\left(g_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The volume form $dV = \sqrt{\det g_{ij}}\, dr \wedge d\theta \wedge d\varphi = r^2 \sin\theta dr \wedge d\theta \wedge d\varphi$

§5 參考 RG1202stokes

$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$

求 $\int_{S^2} \omega =$ 其中Ω是實心球， $S^2 = \partial\Omega$