§ A way to compute the Christoffel symbols

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

$$I = \frac{1}{2}\int f d\tau = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau$$

Consider the change in the integral under infinitesimal variations of the path。

$$x^\mu \to x^\mu + \delta x^\mu$$

$$g_{\mu\nu} \to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu})\delta x^\sigma$$

$$\delta I = \frac{1}{2}\int [\partial_\sigma g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma + g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau}]d\tau$$

The last terms can be integrated by parts，for example，

$$\frac{1}{2}\int\left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau}\right]d\tau = -\frac{1}{2}\int\left[g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2} + \frac{dg_{\mu\nu}}{d\tau}\frac{dx^\mu}{d\tau}\right]\delta x^\nu\, d\tau$$

$$= -\frac{1}{2}\int\left[g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2} + \partial_\sigma g_{\mu\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau}\right]\delta x^\nu\, d\tau,$$

where we have neglected boundary terms, which vanish because we take our variation δx^μ to vanish at the endpoints of the path. In the second line we have used

the chain rule on the derivative of $g_{\mu\nu}$. The variation (3.51) then becomes, after rearranging some dummy indices,

$$\delta I = -\int\left[g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]\delta x^\sigma\, d\tau.$$

Since we are searching for stationary points, we want δI to vanish for any variation δx^σ; this implies

$$g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \qquad (3.54)$$

and multiplying by the inverse metric $g^{\rho\sigma}$ finally leads to

$$\frac{d^2x^\rho}{d\tau^2} + \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \qquad (3.55)$$

We see that this is precisely the geodesic equation (3.40), but with the specific choice of Christoffel connection (3.27). Thus, on a manifold with metric, extremals of the length functional are curves that parallel transport their tangent vector with respect to the Christoffel connection associated with that metric. It doesn't matter if any other connection is defined on the same manifold. Of course, in GR the Christoffel connection is the only one used, so the two notions are the same.

The variational principle provides a convenient way to actually calculate the Christoffel symbols for a given metric. Rather than simply plugging into (3.27), it is often less work to explicitly vary the integral (3.49), with the metric of interest substituted in for $g_{\mu\nu}$. An example of this procedure is shown in Section 3.5.

以上是[Spacetime and Geometry]中 Ch3 Curvature 練習 Variation 的運算。

§ 用 Euler-Lagragain equation 算出 geodesic，由此找到 Christoffel symbols