§ 1 Divergence theorem to Green identity

Divergence theorem in R^n ， $\int_{\Omega}(\nabla\cdot F)dV=\int_{\partial\Omega}F\cdot ndS\cdots(*)$

Green first identity：

$$\int_{\Omega}(u\Delta v+\nabla u\cdot\nabla v)d\Omega=\int_{\partial\Omega}u\frac{\partial v}{\partial n}dS$$

Let $F=u\Delta v$ then divergence $\nabla\cdot F=\nabla\cdot(u\nabla v)=\nabla u\cdot\nabla v+u\Delta v$

代入(*) 右式= $\int_{\partial\Omega}F\cdot ndS=\int_{\partial\Omega}(u\nabla v)\cdot ndS=\int_{\partial\Omega}u\frac{\partial v}{\partial n}dS$

舉例說明：

Given two scalar function u(x)，v(x)， $x\in R^n$

$F=u\nabla v$

For n=3， $\nabla v=(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial v}{\partial z})$ ， $F=(u\frac{\partial v}{\partial x},u\frac{\partial v}{\partial y},u\frac{\partial v}{\partial z})$ then

$$\nabla\cdot F=\frac{\partial}{\partial x}(u\frac{\partial v}{\partial x})+\frac{\partial}{\partial y}(u\frac{\partial v}{\partial y})+\frac{\partial}{\partial z}(u\frac{\partial v}{\partial z})$$

$$=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+\ldots$$

$$=(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z})+u(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})=u\nabla v\cdot\nabla u$$

Green second identity：

$$\int_{\Omega}(u\Delta v-v\Delta u)dV=\int_{\partial\Omega}(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n})dS$$

舉例驗證：

$u=x^2+y^2,v=z^2,\Omega:x^2+y^2+z^2\leq 1$

…

則左=右= $-\frac{4\pi}{5}$

Note that spherical coordinates $x=\sin\theta\cos\phi,y=\sin\theta\sin\phi,z=\cos\theta$ then

$dS=\sin\theta d\theta d\phi$

§ 2 from R^3 to Riemannian manifold (M,g)

In R^3 ， $divX=\partial_i X^i=\frac{\partial X^1}{\partial x}+\frac{\partial X^2}{\partial y}+\frac{\partial X^3}{\partial z}$

In (M,g)， $divX=\nabla_i X^i=\partial_i X^i+\Gamma^i_{ik}X^k=\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}X^i)$ ，where $\Gamma^i_{ik}=\frac{1}{\sqrt{|g|}}\partial_k\sqrt{|g|}$

Divergence theorem：

$\int_M divX\,\mu = \int_{\partial M} \iota_X \mu$，where μ is the Riemannian volume form。

If $\omega = \iota_X \mu$ then $d\omega = d(\iota_X \mu) = (divX)\mu$，the divergence theorem is $\int_M d\omega = \int_{\partial M} \omega$

(Stokes theorem)
Lie derivative of the volume form $L_X \mu = (divX)\mu$

§ Laplacian Δ

In R^3，$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

In (M,g)，$\Delta := div(grad)$

§ For a Killing vector field X，by definition $L_X g = 0$

In local coordinates $L_X g_{\mu\nu} = \nabla_\mu X^\nu + \nabla_\nu X^\mu = 0$

(The Lie derivative of the metric along a vector field X with componens X^k is given by：

$(L_X g)_{ij} = X^k \partial_k g_{ij} + g_{ik}\partial_j X^k + g_{jk}\partial_i X^k$)

Take $\mu = \nu$ then $\nabla_\mu X^\mu = 0$ i.e.div(X)=0

Killing 向量場ξ^μ滿足 Killing 方程$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$，這表示該向量場對應於一種等距變換(isometry)，即時空中的測地距離不變的變換。
因此 Killing vector field 描述了時空的對稱性。例如

(1) A timelike Killing vector (∂_t) generates time translations。
(2) A spacelike Killing vector (e.g., $\partial\phi$) generates spatial rotations，linked to angular momentum conservation。

§ 3 線性代數

X is a vector field。$A_W : X \to \nabla_X W$，where $\nabla_X W$ is the covariant derivative of W along X。

$divX = tr(\nabla X)$，where ∇X is the Jacobian matrix。

Then $tr(A) = \frac{n}{\omega_{n-1}} \int_{S^{n-1}} < A(X), X > dS$，where S^{n-1} is a unit sphere，and ω_{n-1} is the surface area of S^{n-1}。

Let $\{e_i\}$ be an orthonormal basis for R^n。

The trace of A is $tr(A)=\sum_{i=1}^{n}\langle \mathrm{A}(e_i),e_i\rangle$ 。The unit sphere S^{n-1} is symmetric，so the average of $\langle A(X),X\rangle$ over all unit vector X is proportional to tr(A)。

Specially： $\int_{S^{n-1}}\langle A(X),X\rangle dS=C\cdot tr(A)$ ，where C is a constant dependind only on n。

To find C，let A=I(the identity operator)。Then $\langle A(X),X\rangle=\langle X,X\rangle=1$

The integral becomes： $\int_{S^{n-1}}dS=\omega_{n-1}$ 。For A=I，tr(A)=n，so $\omega_{n-1}=C\cdot n\Rightarrow C=\frac{\omega_{n-1}}{n}$

Substitute C back， $\int_{S^{n-1}}\langle A(X),X\rangle dS=\frac{\omega_{n-1}}{n}tr(A)$

$$tr(A)=\frac{n}{\omega_{n-1}}\int_{S^{n-1}}\langle A(X),X\rangle dS$$

舉例說明：

1. Let $W(x,y,z)=(x^2,y,z)$

Jacobian $J_W=\begin{bmatrix}2x&0&0\\0&1&0\\0&0&1\end{bmatrix},tr(\mathrm{J}_W)=2x+2$

Operator A： $A(X)=J_W\cdot X=2xX^1+X^2+X^3$

$$\langle A(X),X\rangle=2xX_1^2+X_2^2+X_3^2$$

Using symmetry，average over S^2 ： $\int_{S^2}X_i^2dS=\frac{4\pi}{3}$ for i=1,2,3

So $\int_{S^2}\langle A(X),X\rangle dS=\frac{4\pi}{3}(2x+2)$

$$\frac{n}{\omega_{n-1}}\int_{S^{n-1}}\langle A(X),X\rangle dS=\frac{3}{4\pi}\cdot\frac{4\pi}{3}(2x+2)=2x+2=tr(J_W)$$

The derivative matrix ∇X (Jacobian) is the backbone of the problem。

1. It defines the operator A via $A(X)=\nabla_X W$.
2. Its trace gives the divergence, which is averaged over the sphere.
3. The integral identity elegantly connects local (trace) and global (integral) properties of A.

2. F(x,y,z)=(yz,xz,xy)，compute and verify $tr(A)=\frac{3}{4\pi}\int_{S^2}\langle A(X),X\rangle dS$

Where X= (X_1,X_2,X_3) is a unit vector on S^2 ， $A(X)=\nabla_X F=J_F\cdot X$

1. $J_F = \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}$ ， $tr(A) = tr(J_F) = 0$

2. $A(X) = \nabla_X F = J_F \cdot X = \begin{pmatrix} zX_1 + yX_3 \\ zX_1 + xX_3 \\ yX_1 + xX_2 \end{pmatrix}$

   $\langle A(X), X \rangle = ... = 2zX_1X_2 + 2yX_1X_3 + 2xX_2X_3$

3. Terms like X_1X_2 are odd functions over S^2 ，so $\int_{S^2} X_1X_2 dS = ... = 0$

   $\int_{S^2} \langle A(X), X \rangle dS = 0$

The field $\mathbf{F}(x,y,z) = (yz, xz, xy)$ is **curl-free** ($\nabla \times \mathbf{F} = \mathbf{0}$) and divergence-free, making it a **harmonic vector field** in $\mathbb{R}^3$. Such fields often arise in solutions to Laplace's equation $\nabla^2 \mathbf{F} = \mathbf{0}$.

Try repeating this for $\mathbf{F}(x,y,z) = (-y, x, 0)$ (a rotational field) to see non-zero curl and a different trace!

§

in R^3 ， $divX = \partial_i X^i = \frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y} + \frac{\partial X^3}{\partial z}$

in (M,g)， $divX = \partial_i X^i + \Gamma^i_{ik} X^k = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} X^i)$ ，where $\Gamma^i_{ik} = \frac{1}{\sqrt{|g|}} \partial_k \sqrt{|g|}$

這與測地線方程類似

直線 $\ddot{x}^k = 0$

geodesic： $\ddot{x}^k + \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0$