§ 向量場奇點的標數

1. 切線轉角(Hopf Umlaufssatz)定理
2. Holonomy 角 $\delta_C\alpha = \int_\Omega KdM$ ，$IndX = \frac{1}{2\pi}\delta_C\varphi$
3. Gauss-Bonnet 定理
4. Hopf-Poincare 定理 奇點標數和= $\chi(M)$
5. $\sum_{p_i} Ind_{p_i} X = \frac{1}{2\pi}\int_M KdM = \chi(M)$

Definition 3.1 A point $p \in M$ is said to be a **singular point** of X if $X_p = 0$. A singular point is said to be an **isolated singularity** if there exists a neighborhood $V \subset M$ of p such that p is the only singular point of X in V.

Since M is compact, if all the singularities of X are isolated then they are in finite number (as otherwise they would accumulate on a non-isolated singularity).

To each isolated singularity $p \in V$ of $X \in \mathfrak{X}(M)$ one can associate an integer number, called the **index** of X at p, as follows:

(i) fix a Riemannian metric in M;

(ii) choose a positively oriented orthonormal frame $\{F_1, F_2\}$, defined on $V\backslash\{p\}$, such that

$$F_1 = \frac{X}{\|X\|},$$

let $\{\overline{\omega}^1, \overline{\omega}^2\}$ be the dual coframe and let $\overline{\omega}_1^2$ be the corresponding connection form;

(iii) possibly shrinking V, choose a positively oriented orthonormal frame $\{E_1, E_2\}$, defined on V, with dual coframe $\{\omega^1, \omega^2\}$ and connection form ω_1^2;

(iv) take a neighborhood D of p in V, homeomorphic to a disc, with a smooth boundary ∂D, endowed with the induced orientation, and define the index I_p of X at p as

$$2\pi I_p = \int_{\partial D} \sigma,$$

where $\sigma := \overline{\omega}_1^2 - \omega_1^2$ is the form in Proposition 2.7.

Recall that σ satisfies $\sigma = d\theta$, where θ is the angle between E_1 and F_1. Therefore I_p must be an integer. Intuitively, the index of a vector field X measures the number of times that X rotates as one goes around the singularity anticlockwise, counted positively if X itself rotates anticlockwise, and negatively otherwise.

Example 3.2 In $M = \mathbb{R}^2$ the following vector fields have isolated singularities at the origin with the indicated indices (cf. Fig. 4.1):

(1) $X_{(x,y)} = (x, y)$ has index 1;
(2) $Y_{(x,y)} = (-y, x)$ has index 1;
(3) $Z_{(x,y)} = (y, x)$ has index -1;
(4) $W_{(x,y)} = (x, -y)$ has index -1.

Theorem 3.3 (Gauss–Bonnet) *Let M be a compact, oriented, 2-dimensional manifold and let X be a vector field in M with isolated singularities $p_1, \ldots, p_k$. Then*

$$\int_M K = 2\pi \sum_{i=1}^{k} I_{p_i} \tag{4.7}$$

for any Riemannian metric on M, where K is the Gauss curvature.

例 求 X=(x^3-3xy^2, y^3-3x^2y)在(0，0)的標數 do Carmo p.283

$I=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{dt}dt$ ，其中φ是向量場 v 與 X_u 的夾角

$v=(x^3-3xy^2, y^3-3x^2y)$ ， $x=\cos t, y=\sin t$

$|v|^2=(\cos^3 t-3\cos t\sin^2 t)^2+(\sin^3 t-3\cos^2 t\sin t)^2=\cos^6 t+\sin^6 t+3\sin^2 t\cos^2 t=\ldots=1$

$|v|=1$ ，所以設 $\cos^3 t-3\cos t\sin^2 t=\cos\varphi, \sin^3 t-3\cos^2 t\sin t=\sin\varphi$

則

$-\sin\varphi\frac{d\varphi}{dt}=-3\sin t\cos^2 t+3\sin^3 t-6\sin t\cos^2 t=-3\sin t(3\cos^2 t-\sin^2 t)$ ...(1)

$\cos\varphi\frac{d\varphi}{dt}=3\sin^2 t\cos t-3\cos^3 t+6\sin^2 t\cos t=3\cos t(3\sin^2 t-\cos^2 t)$ ...(2)

$(\frac{d\varphi}{dt})^2=9\sin^2 t(9\cos^4 t+\sin^4 t-6\sin^2 t\cos^2 t)$

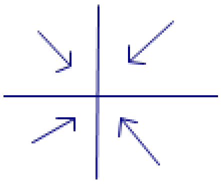
$+9\cos^2 t(9\sin^4 t+\cos^4 t-6\sin^2 t\cos^2 t)$

$=81\sin^2 t\cos^2 t+9(\sin^6 t+\cos^6 t)-54\sin^2 t\cos^2 t$

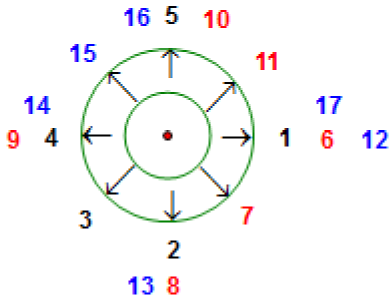
$=27\sin^2 t\cos^2 t+9(1-3\sin^2 t\cos^2 t)=9$

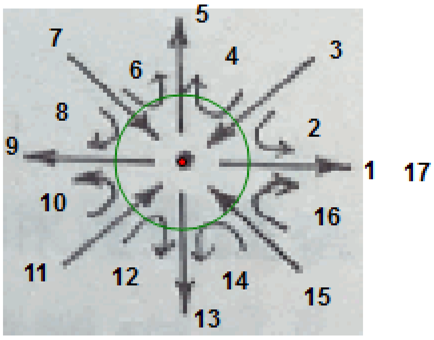
所以 I=± 3

$V_{(1,1)}=(-2,-2), V_{(-1,1)}=(2,-4), V_{(-1,-1)}=(2,2), V_{(1,-1)}=(-2,4)$

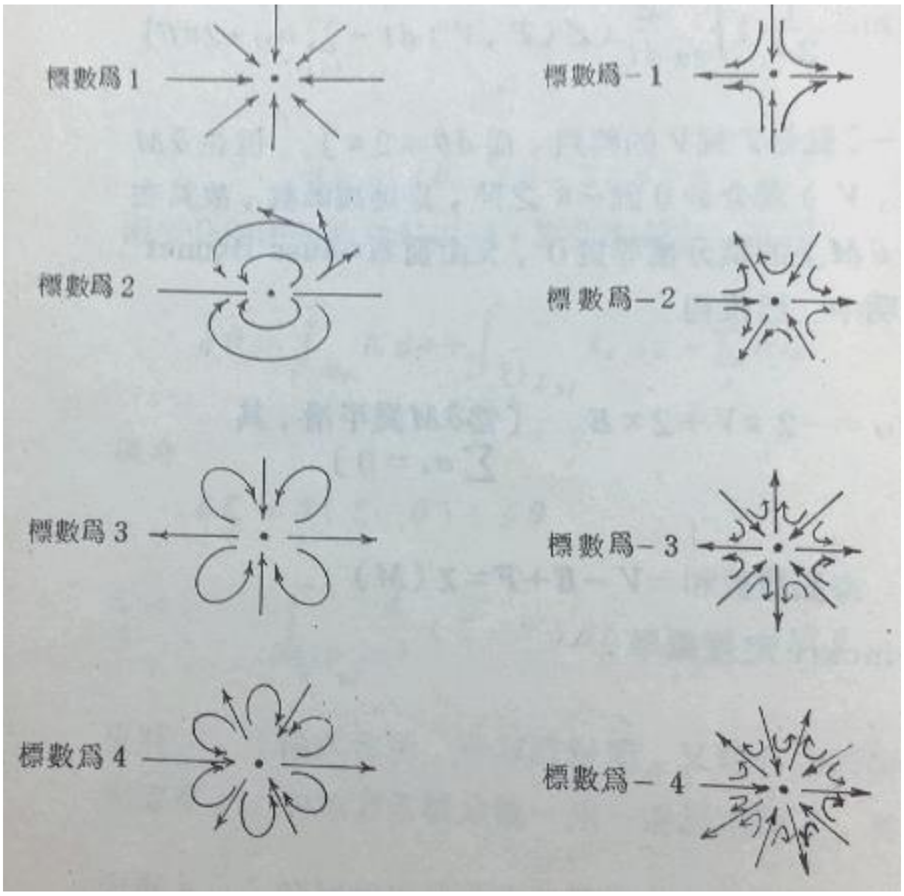
圖長這樣  ，所以 I=-3，

所以原向量場的水流圖就是下圖





繞奇點逆時針轉一圈，水流的切向量順時針轉 3 圈，所以 I=-3
奇點標數與附近水流的情況。



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$$\int_M K = 2\pi \sum_{i=1}^{k} I_{p_i} \tag{4.7}$$

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The prove is in p.144

1. 初等微分幾何講稿　黃武雄　p.151 Hopf-Poincare 標數定理
2. 大域微分幾何　p.30
3. An Introduction to Riemannian Geometry p.142 Gauss-Bonnet theorem
4. Differential Geometry of Curves and Surface　P.do Carmo　P.282 Exercise 4-5 (6)