§ 完備的向量場

$\varphi_t : W \to M$

(1) Local diffeomorphism

(2) $(\varphi_t \circ \varphi_s)(q) = \varphi_{t+s}(q)$ ---(*)

$\{\varphi_t : I \to M\}_{t\in I}$ ， $I = (-\varepsilon, \varepsilon)$ 滿足(*) 稱為 local one-parameter group (of diffeomorphism)

一個向量場 X，其 local flow 定義一個 one-parameter group of diffeomorphism，

$\left(\varphi_t \circ \varphi_s\right)(q) = \varphi_{t+s}(q)$ ， $\varphi_0 = identity$

(all solution curves exist for all time。換句話說 flow 定義到整個 R)

則稱 X 為完備的(complete)向量場。

一個 one-parameter group 的軌跡(orbit)是向量場的積分曲線(a curve tangent to vector field)

例

1. $X \in \chi(R)$ ， $X = x^2 \dfrac{d}{dx}$ is incomplete

2. $X = x\dfrac{\partial}{\partial x}$ on M=R is complete

3. M=R² $X_1 = \dfrac{\partial}{\partial x}$ ， $X_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$ ， $X_3 = y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$ are all complete

解

1. $X = x^2 \dfrac{d}{dx}$ ，求 $\dot{x} = x^2$ 的積分曲線

$\dfrac{dx}{dt} = x^2$ ， $\dfrac{dx}{x^2} = dt$ 兩邊積分 $-\dfrac{1}{x} = t + c$ ， $x = \dfrac{-1}{t+c}$ $\therefore \varphi_t = \dfrac{-1}{t+c}$ ， $\varphi_0(x) = x$

$-\dfrac{1}{c} = x$ $\therefore \varphi_t = \dfrac{x}{1-tx} : W \to M$

設 W=(a,b)，a>0

$F : W \times I \to M$ F(q,0)=q， $\dfrac{\partial F}{\partial t}(q,t) = X_{F(q,t)}$ 則 local flow 只能延伸到 $W \times (-\infty, \dfrac{1}{b})$

另一種說法：

For initial condition $x(0) = x_0 \neq 0$ ， $x(t) = \dfrac{x_0}{1-tx_0}$ 在 $t = \dfrac{1}{x_0}$ 沒有定義。

所以 X 非完備向量場。

力學中 大部分的(Hamiltonian)向量場不是完備的向量場。

3. $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ on R^2

求 X 的 flow,積分曲線 並且說明 X 是完備向量場

$\varphi_t = (\varphi_1(t), \varphi_2(t))$

$\dot{\varphi}_1(t) = \varphi_1(t)$ ， $\varphi_1(t) = ae^t$

$\dot{\varphi}_2(t) = \varphi_2(t)$ ， $\varphi_2(t) = be^t$

$\because \varphi_0(x,y) = (x,y)$ ，所以 a=x，b=y

$\varphi_t(x,y)=(xe^t, ye^t)$ ， $\varphi_{s+t} = \varphi_s \circ \varphi_t$ 是 1-parameter group with $\varphi_0 = identity$

定理

Let X,Y 是 M 上的完備光滑向量場 with flows φ, ψ

φ, ψ commute $\Leftrightarrow [X,Y]=0$

命題

Every vector field on a compact manifold is complete。

習作

[D01] p.33 Ans at p.333

Let $X, Y \in \mathfrak{X}(M)$ be two complete vector fields with flows ψ, ϕ. Show that:

(a) given a diffeomorphism $f : M \to M$, we have $f_* X = X$ if and only if $f \circ \psi_t = \psi_t \circ f$ for all $t \in \mathbb{R}$;

(b) $\psi_t \circ \phi_s = \phi_s \circ \psi_t$ for all $s, t \in \mathbb{R}$ if and only if $[X, Y] = 0$.

解

(b)

$\Rightarrow$ $\psi_t \circ \phi_s = \phi_s \circ \psi_t$ for $\forall s,t \in R$

By (a) $(\psi_t)_* Y = Y$ ， $[X,Y] = L_X Y = \frac{d}{dt}((\psi_{-t})_* Y)\big|_{t=0} = \frac{d}{dt} Y\big|_{t=0} = 0$

If, on the other hand, $[X, Y] = 0$ then

$$\frac{d}{dt}((\psi_t)_* Y) = \frac{d}{d\varepsilon}((\psi_{t+\varepsilon})_* Y)_{|\varepsilon=0} = \frac{d}{d\varepsilon}((\psi_t)_*(\psi_\varepsilon)_* Y)_{|\varepsilon=0}$$
$$= (\psi_t)_* \frac{d}{d\varepsilon}((\psi_\varepsilon)_* Y)_{|\varepsilon=0} = -(\psi_t)_* L_X Y = 0.$$

Since $(\psi_0)_* Y = Y$, we conclude that $(\psi_t)_* Y = Y$ for all $t \in \mathbb{R}$. Therefore $\psi_t \circ \phi_s = \phi_s \circ \psi_t$ for all $s, t \in \mathbb{R}$.

習作
[DG05] p.117

EX 10

Show that the vector field on $\boldsymbol{R}^2$

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$$

gives rise to a one-parameter (global) group of diffeomorphisms that will be specified.

Answer. The vector field may be interpreted as the right-hand side of the following system:

$$\frac{dx}{dt}=x \qquad \frac{dy}{dt}=y.$$

Given the initial conditions

$$x(0)=x_0 \quad \text{and} \quad y(0)=y_0,$$

the differential system admits an integral curve

$$c:R\to\boldsymbol{R}^2:t\mapsto(x(t),y(t))$$

defined by

$$x(t)=x_0e^t \qquad y(t)=y_0e^t$$

and through (x_0,y_0) at initial "instant."
The field is complete because each integral curve is defined to each "instant."
The integral curves are semi-straight line without origin.

The one-parameter group is made up of the following diffeomorphisms:

$$\phi_t:\boldsymbol{R}^2\to\boldsymbol{R}^2:(x_0,y_0)\mapsto\phi_t(x_0,y_0)=(x(t),y(t))=(x_0e^t,y_0e^t).$$

Since

$$\phi_t(x_0,y_0)=e^t(x_0,y_0),$$

we conclude that ϕ_t is a homothety (center o, ratio e^t).

例

ϕ:$\mathrm{R}\times\mathrm{R}^2\to R^2$，(t,(x,y))$\to$(x+t,y-3t)

是否定義一個 one-parameter group？

$\phi(t,u)=\varphi_t(u)$ 則

$\varphi_0(x,y)=(x,y)\Rightarrow\varphi_0=id$

$\varphi_t(x,y)=(x+t,y-3t)$

驗證 $\varphi_s\circ\varphi_t(x,y)=\ldots=\varphi_{s+t}(x,y)$，

$(\varphi_s^{-1}\circ\varphi_s)(x,y)=\varphi_s^{-1}(x+s,y-3s)=(x,y)$，$\therefore\varphi_s^{-1}(x,y)=(x-s,y+3s)=\varphi_{-s}(x,y)$

所以 ϕ 定義一個 one-parameter group

例

$\phi: R\times R^2 \to R^2$

$\phi(t,(x,y))=(tx, y-x)$ 沒定義一個 one-parameter group

因為 $\varphi_t(x,y)=(tx,y-x)$ 則 $\varphi_0(x,y)=(0,y-x)\neq(x,y)$

例

$\phi: \mathrm{R}\times\mathrm{R}^2 \to R^2$

$\phi(t,(x,y))=(x\cos 2t - y\sin 2t, x\sin 2t + y\cos 2t + t)$ 沒定義一個 one-parameter group

$\varphi_t(x,y)=(x\cos 2t - y\sin 2t, x\sin 2t + y\cos 2t + t)$

$\varphi_0(x,y)=(x,y)$

但是 $\varphi_{s+t}\neq\varphi_s\circ\varphi_t$ (要驗證)

習作 14

Prove the mapping $\phi: R\times R^2 \to R^2 : (t,(x,y)) \to (x\cos t - y\sin t, x\sin t + y\cos t)$ defines a one-parameter group。What is this group？

Plane rotation group

例 $\mathrm{X}=x\dfrac{\partial}{\partial y}-y\dfrac{\partial}{\partial x}$

(1) 求 X 的 flow
(2) 求 X 的積分線
(3) 說明 X 是完備向量場

φ_t is the flow of $X=x\dfrac{\partial}{\partial y}-y\dfrac{\partial}{\partial x}$ with $\dfrac{d\varphi_t(q)}{dt}=X_{\varphi_t(q)}$ and $\varphi_0(q)=q$

$$\begin{cases}\dot{x}=-y\\ \dot{y}=x\end{cases} \text{ with } x(0)=x_0, y(0)=y_0$$

$\ddot{y}=\dot{x}=-y$ then $y(t)=x_0\sin t+y_0\cos t$ ， $x(t)=x_0\cos t-y_0\sin t$

$$\varphi_t(x,y)=(x_0\cos t-y_0\sin t, x_0\sin t+y_0\cos t)=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix}$$

The vector field is complete。

The integral curves are concentric circles $x^2+y^2={x_0}^2+{y_0}^2$ 。

$\{\varphi_t | t \in R\} = SO(2,R)$ 是一個旋轉群

Exercise 17. (Important!)

With the help of diffeomorphisms give an interpretation of the Lie derivative of a vector field X with respect to vector field Y on a manifold M.
Given one-parameter groups of diffeomorphisms ϕ_t and ψ_t, of which X and Y are the respective generating fields, show that the curve

$$t \mapsto (\phi_{-\sqrt{t}} \circ \psi_{-\sqrt{t}} \circ \phi_{\sqrt{t}} \circ \psi_{\sqrt{t}})x$$

is differentiable at $t = 0$ and admits $[X, Y] = L_X Y$ as a corresponding tangent vector.

§ V.I. Arnold

$\frac{d\varphi}{dt} = V(t, \varphi(t))$ is the phase velocity vector field，then the phase flow is $\varphi_t(x)$

with $\varphi_0(q) = q$

例

1. $\dot{x} = kx$

$\dot{\varphi} = k\varphi$ with $\varphi_0 = identity$，then $\varphi_t(x) = xe^{kt}$，the phase flow is the group

$\{xe^{kt}\}$，which is a one-parameter diffeomorphism group。

參考[https://jmath2020.neocities.org/DifferentialEquation/DEflows.pdf]
關於一個微分方程的 flow

一個 vector field 與一個微分方程應該是指同一現象。

2. Show that the vector field $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ is complete on R^2。

$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ with $x(0) = x_0, y(0) = y_0$，then $\ddot{y} = \dot{x} = -y$

$y(t) = x_0 \sin t + y_0 \cos t, x(t) = x_0 \cos t - y_0 \sin t$

$(x(t))^2 + (y(t))^2 = x_0^2 + y_0^2$ the integral curves are concentric circles。

The one-parameter group：

$\varphi_t(x,y)=(x\cos t-y\sin t, x\sin t+y\cos t), \varphi_0(x,y)=(x,y)$

$\{\varphi_t \mid t\in \mathrm{R}\}$=SO(2,R)

§ $X=xz\frac{\partial}{\partial x}+yz\frac{\partial}{\partial y}-(x^2+y^2)\frac{\partial}{\partial z}$ ，求通過 A(2,-1,1)的 integral curve

$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{-(x^2+y^2)}$

由 $\frac{dx}{x}=\frac{dy}{y}\Rightarrow y=c_1x$

$\frac{xdx}{x^2}=\frac{ydy}{y^2}=\frac{dz}{-(x^2+y^2)}$ ， $\frac{d(x^2+y^2)}{2(x^2+y^2)}=\frac{dz}{-(x^2+y^2)}\Rightarrow x^2+y^2+z^2=c_2$

A 點代入 得

$$C:\begin{cases} y=-\frac{1}{2}x \\ x^2+y^2+z^2=6 \end{cases}$$

$P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}=R$ is a Lagrange linear equation，then $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

§ $\dot{X}=AX$

$\exp:\eta\to G$ with $V\to\varphi_1(e)$

其中 φ_t 是 left-invariant vector field X^V 的 flow，then $\varphi_t(e)=\exp(tV)$

Consider $h(t)=e^{tA}$ ，then h(0)=I

$\frac{d}{dt}h(t)=Ae^{tA}=Ah(t)$ ，so h(t) is the flow of X^A at e (i.e. $h(t)=\varphi_t(e)$)

Then $\exp A=\varphi_1(e)=e^A$

$\exp A=\sum_{k=0}^{\infty}\frac{1}{k!}A^k$ ， $M_{n\times n}(R)\overset{\exp}{\to}GL(n,R)$

$\varphi_A(t)=\exp(tA)$ is a one-parameter group with $\frac{d}{dt}(\exp tA)\big|_{t=0}=A$

So $M_{n\times n}(R)$ is the Lie algebra of GL(n ,R) with [ ]。