N1101Vector fields

§ 1.1 Definition of a vector on a manifold M

M 是一C^∞流形， $p \in M$ ，設 U 是 p 的一個鄰域。

Smooth function $X_p : C^\infty(U) \to R$ 滿足

1. 線性 X(a f +b g)=a(X f)+b(X g)
2. $X_p(fg) = (X_p f)g(p) + f(p)(X_p g)$

這樣的線性函數X_p稱為 M 上在 p 點的切向量。

亦即 Smooth 切向量 X 是從$C^\infty(M)$到$C^\infty(M)$的算子。

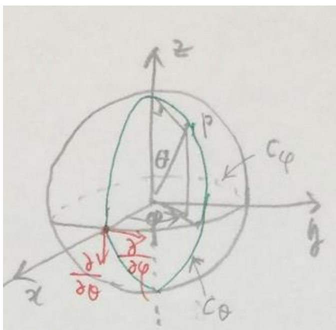
若 $f \in C^\infty(M)$則$(Xf)(p) = X_p f$ 。

例如

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial x},\ f(x,y,z) = 2x^2y + z \text{ 則 } X(f) = y - z(4xy) = y - 4xyz$$

§ 1.2 過 p 點的切向量全體稱為切空間，記為T_pM 。

$T_pM = \left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$是一個 n 維的向量空間。



例 S^2

$\psi : (0,\pi)\times(-\pi,\pi) \to S^2$ given by

$\psi(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

Parameterizes a neighorhood of the point

$(1,0,0) = \psi(\frac{\pi}{2}, 0)$

Consequently，$\left(\frac{\partial}{\partial\theta}\right)_{(1,0,0)} = \dot{c}_\theta(0), \left(\frac{\partial}{\partial\varphi}\right)_{(1,0,0)} = \dot{c}_\varphi(0)$ ，where

$c_\theta(t)=\psi(\frac{\pi}{2}+t,0)=(\cos t,0,-\sin t)$ ； $c_\varphi(t)=\psi(\frac{\pi}{2},t)=(\cos t,\sin t,0)$

Since c_θ and c_φ are curves in R^3，$\left(\frac{\partial}{\partial\theta}\right)_{(1,0,0)}$ and $\left(\frac{\partial}{\partial\varphi}\right)_{(1,0,0)}$ can be identified with the vectors (0,0,-1) and (0,1,0)

§ 1.3 flow of a vector field and Lie derivative of a vector field

$\varphi:U\to M \quad \varphi_t(p)=\gamma(t), t\in(-\varepsilon,\varepsilon)$，where $\frac{d(\varphi_t(p))}{dt}=\frac{d\gamma}{dt}=X$，then φ_t is called the flow of vector field X。

φ_t is a 1-parameter group，$\varphi_t\circ\varphi_s(q)=\varphi_{t+s}(q)$，$\varphi_0=identity$

Y is a C^∞ vector field，the Lie derivative of Y along X is $L_XY=\lim_{t\to0}\frac{(\varphi_{-t})_*Y-Y}{t}$

$[X,Y]=\sum_i(XY^i-YX^i)\frac{\partial}{\partial x^i}$，then $L_XY=[X,Y]$

例

1. $X=-(x^2+y^2+z^2)^{-\frac{3}{2}}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})$，$Y=\frac{\partial}{\partial y}, r=(x^2+y^2+z^2)^{\frac{1}{2}}$

求 $L_XY=$

$$L_XY=\sum_i(XY^i-YX^i)\frac{\partial}{\partial x^i}$$

$$XY^1-YX^1=\frac{\partial}{\partial y}((x^2+y^2+z^2)^{-\frac{3}{2}}x)=-3xyr^{-5}$$

$$XY^2-YX^2=\frac{\partial}{\partial}((x^2+y^2+z^2)^{-\frac{3}{2}}y)=-3r^{-5}y^2+r^{-3}=r^{-5}(r^2-3y^2)$$

$$XY^3-YX^3=\frac{\partial}{\partial y}((x^2+y^2+z^2)^{-\frac{3}{2}}z)=-3yzr^{-5}$$

所以 $L_XY=r^{-5}\{-3xy\frac{\partial}{\partial x}+(r^2-3y^2)\frac{\partial}{\partial y}-3yz\frac{\partial}{\partial z}\}$

2. $X=y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}$，$Y=z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}$

(1) 求 X 的 flow φ_t，Y 的 flow ψ_t

(2) 求[X,Y]=

(3) 驗證 $L_X Y = \lim_{t\to 0}\frac{(\varphi_{-t})_* Y - Y}{t} = [X,Y]$

(1) $\frac{d\varphi_t(p)}{dt} = X_{\varphi_t(p)}, \varphi_{0(p)} = p$

$$\begin{cases} \dot{\varphi}_t^1(p) = X^1(\varphi_{t(p)}) = 0 \\ \dot{\varphi}_t^2(p) = X^2(\varphi_{t(p)}) = -\varphi_t^3 \quad \Rightarrow \varphi_t^1 = c, \ddot{\varphi}_t^2 = -\dot{\varphi}_t^3 = -\varphi_t^2 \\ \dot{\varphi}_t^3(p) = X^3(\varphi_{t(p)}) = \varphi_{t(p)}^2 \end{cases}$$

Then $\varphi_t^1 = C,\ \varphi_t^2 = A\cos t + B\sin t, \varphi_t^3 = A\sin t - B\cos t$

A，B，C are function of p=(x y z)

And $\varphi_0(x,y,z) = (x,y,z)$ ，so C=x，A=y，B=-z

$$\varphi_t(x,y,z) = (x, y\cos t - z\sin t, y\sin t + z\cos t)$$

$\varphi_{-t}(x,y,z) = (x, y\cos t + z\sin t, -y\sin t + z\cos t) = (\hat{x},\hat{y},\hat{z})$ ...(*)

同理

$$\psi_t(x,y,z) = (x\cos t + z\sin t, y, -x\sin t + z\cos t)$$

(2) $[X,Y] = (XY^i - YX^i)\frac{\partial}{\partial^i} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

(3) $(\varphi_{-t})_* Y = Y$ (in coordinate $\hat{x},\hat{y},\hat{z}$)

由(*)解出 $x = \hat{x}, y = \hat{y}\cos t - \hat{z}\sin t, z = \hat{y}\sin t + \hat{z}\cos t$

$$Y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} = (\hat{y}\sin t + \hat{z}\cos t)\frac{\partial}{\partial \hat{x}} - \hat{x}(\sin t\frac{\partial}{\partial \hat{y}} + \cos t\frac{\partial}{\partial \hat{z}})\Big|_{at(\hat{x},\hat{y},\hat{z})}$$

Then $L_X Y = \frac{d}{dt}((\varphi_{-t})_* Y)_{t=0}$

$$= (\hat{y}\cos t - \hat{z}\sin t)\frac{\partial}{\partial \hat{x}} - (\hat{x}\cos t)\frac{\partial}{\partial \hat{y}} + (\hat{x}\sin t)\frac{\partial}{\partial \hat{z}}\Big|_{t=0} = \hat{y}\frac{\partial}{\partial \hat{x}} - \hat{x}\frac{\partial}{\partial \hat{y}}$$

Exercises

(1) $X=-(x^2+y^2+z^2)^{-\frac{3}{2}}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})$ ，$Y=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$ ，find $L_X Y=$ 0

(2) $X=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+x(y+1)\frac{\partial}{\partial z}$ ，$Y=\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}$ 求

(a) [X, Y]=

(b) 求 X 的 flow φ_t 並驗證 $\frac{d}{dt}((\varphi_{-t})_* Y)\big|_{t=0}=[X,Y]$

$[X,Y]=(XY^i-YX^i)\partial_i$

$=(XY^1-YX^1)\frac{\partial}{\partial x}+(XY^2-YX^2)\frac{\partial}{\partial y}+(XY^3-YX^3)\frac{\partial}{\partial z}$

($X^i=(1,1,x(y+1)),Y^i=(1,0,y)$

$XY^1=X(1)=0,YX^1=Y(1)=0$

$XY^2=X(0)=0,YX^2=Y(1)=0$

$XY^3=X(y)=1,YX^3=Y(x(y+1))=y+1$

所以[X,Y]= $-y\frac{\partial}{\partial z}$

以下求 X 的 flow φ_t ，並驗證 $\frac{d}{dt}((\varphi_{-t})_* Y)\big|_{t=0}=[X,Y]$

$\dot{\varphi}_t^1=1,\dot{\varphi}_t^2=1$， $\varphi_0(x,y,z)=(x,y,z)$ 所以 $\varphi_t^1=t+x,\varphi_t^2=t+y$

$\dot{\varphi}_t^3=(t+x)(t+y+1),\varphi_0^3=z$ ，所以 $\varphi_t^3=\frac{1}{3}t^3+\frac{1}{2}(x+y+1)t^2+x(y+1)t+z$

$\varphi_{-t}(x,y,z)=(-t+x,-t+y,-\frac{1}{3}t^3+\frac{1}{2}(x+y+1)t^2-x(y+1)t+z)=(\bar{x},\bar{y},\bar{z})$

$(\varphi_{-t})_* Y=Y$ (in coordinate of $\bar{x},\bar{y},\bar{z}$)，

由 chain rule

$\partial_x=\frac{\partial \bar{x}}{\partial x}\partial_{\bar{x}}+\frac{\partial \bar{y}}{\partial x}\partial_{\bar{y}}+\frac{\partial \bar{z}}{\partial x}\partial_{\bar{z}}=\partial_{\bar{x}}+[\frac{1}{2}t^2-(y+1)t]\partial_{\bar{z}}$ ， $y+1=\bar{y}+t+1$

$$\partial_z = \frac{\partial \bar{x}}{\partial z}\partial_{\bar{x}} + \frac{\partial \bar{y}}{\partial z}\partial_{\bar{y}} + \frac{\partial \bar{z}}{\partial z}\partial_{\bar{z}} = \partial_{\bar{z}}$$

所以 $(\varphi_{-t})_* Y = \partial_{\bar{x}} + [\frac{1}{2}t^2 - (\bar{y} + t + 1)t]\partial_{\bar{z}} + (\bar{y} + t)\partial_{\bar{z}}$

$$\frac{d}{dt}((\varphi_{-t})_* Y)_{t=0} = -(\bar{y}+1)\partial_{\bar{z}} + \partial_{\bar{z}} = -\bar{y}\,\partial_{\bar{z}} = [X, Y]_{at(\bar{x},\bar{y},\bar{z})}$$

$$\varphi_t(x, y, z) = (t + x, t + y, \frac{1}{3}t^3 + \frac{1}{2}t^2(x + y + 1) + tx(y + 1) + z)$$

3. $X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

(a) Compute $[X_i, X_j]$

(b) Show that span $\{X_1, X_2, X_3\}$ is a Lie subalgebra of $\chi(R^3)$，isomorphic to

(R^3, X)

(c) Compute flows of X_i

(d) $\varphi_{i,\frac{\pi}{2}} \circ \varphi_{j,\frac{\pi}{2}} \neq \varphi_{j,\frac{\pi}{2}} \circ \varphi_{i,\frac{\pi}{2}}$ for $i \neq j$

(e) Calculate $\frac{\partial}{\partial \varphi}(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$

解

(a) $[X_1, X_2] = -X_3, [X_1, X_3] = X_2, [X_2, X_3] = -X_1$

(b) 假設 $V := span\{X_1, X_2, X_3\}$ with [，]

$F: V \to R^3$

$F(aX_1 + bX_2 + cX_3) = (a, -b, c)$ is a bijective

Show that it is a Lie algebra homeomorphism

即 $F([X_i, X_j]) = \ldots = F(X_i) \times F(X_j)$

例如 $F([X_1, X_2]) = F(-X_3) = (0, 0, -1)$

$$F(X_1)\times F(X_2)=(1,0,0)\times(0,-1,0)=(0,0,-1)$$

So F is a homomorphism， $\therefore \mathrm{V}\cong(\mathrm{R}^3,$ X)

(c) $\varphi_{1,t}(x,y,z)=(x, y\cos t - z\sin t, y\sin t + z\cos t)$

$\varphi_{2,t}(x,y,z)=(x\cos t + z\sin t, y, -x\sin t + z\cos t)$

$\varphi_{3,t}(x,y,z)=(x\cos t - y\sin t, x\sin t + y\cos t, z)$

(d) 略

(e) $M_x(r)=-z\dfrac{\partial r}{\partial y}+y\dfrac{\partial r}{\partial z}=-z\times\dfrac{1}{2}(2y)(x^2+y^2+z^2)^{-\frac{1}{2}}+y\times\dfrac{1}{2}(2z)(x^2+y^2+z^2)^{-\frac{1}{2}}=0$