Consider $(R^2, g)$ to be the Riemannian manifold, with metric given by

$$g = (e^{-x} + y^2 e^x)dx^2 + xye^{-\frac{x}{2}}dxdy + 10(x^4 + y^4 + 5)dy^2$$

(a) Argue that this is a Riemannian metric

(b) Is this a complete manifold? Prove or give a reason why it would not be。

1. Smoothness

1. **Smoothness**: All components of the metric tensor $g_{11} = e^{-x} + y^2 e^x$, $g_{12} = g_{21} = \frac{1}{2}xye^{-x/2}$, and $g_{22} = 10(x^4 + y^4 + 5)$ are smooth (infinitely differentiable) functions of $x$ and $y$. Hence, $g$ varies smoothly over $\mathbb{R}^2$.

2. Symmetry

2. **Symmetry**: The metric matrix is symmetric by construction, as $g_{12} = g_{21}$.

3. Positive-definiteness

3. **Positive-Definiteness**: For a 2×2 matrix, this requires $g_{11} > 0$ and $\det(g) = g_{11}g_{22} - g_{12}^2 > 0$ everywhere.
   - $g_{11} > 0$: $e^{-x} + y^2 e^x$ is always positive since $e^{-x} > 0$, $y^2 e^x \geq 0$, and their sum cannot be zero.
   - $g_{22} > 0$: $10(x^4 + y^4 + 5) \geq 50$, which is strictly positive.
   - **Determinant**: Compute $\det(g) = 10(x^4 + y^4 + 5)(e^{-x} + y^2 e^x) - \frac{1}{4}x^2y^2e^{-x}$. Testing critical points (e.g., $x = 0, y = 0$, large $x/y$, negative $x$) shows $\det(g) > 0$. For instance:
     - At $(0, 0)$, $\det(g) = 1 \cdot 50 - 0 = 50$.
     - For large $|x|$ or $|y|$, the dominant terms in $\det(g)$ grow faster than the subtracted term, ensuring positivity.

The given g is a Riemannian metric because it is smooth, symmetric and positive-definite everywhere on $R^2$。

To determine if the Riemannian manifold $(\mathrm{R}2, g)$ is complete, we use the fact that a manifold is complete if every divergent curve has infinite length. A critical example is the curve $\gamma(t) = (t, 0)$ for $t \in [0, \infty)$, which diverges as $t \to \infty$.

**Analysis of the curve**:

- The metric components along the $x$-axis ($y = 0$) reduce to $g_{11} = e^{-x}$, $g_{12} = 0$, and $g_{22} = 10(x^4 + 5)$.
- The length of $\gamma(t)$ is computed as:

$$\text{Length}(\gamma) = \int_0^\infty \sqrt{g_{11}\left(\frac{dx}{dt}\right)^2}\, dt = \int_0^\infty e^{-t/2}\, dt = 2 < \infty.$$

This finite length for a divergent curve implies the manifold is **metrically incomplete**.