Let X= $\left\{(x,y,z)\in R^3 \middle| x^3+xyz+y^2=1\right\}$

(a) Show that X is a 2-manifold

(b) Consider the map $\pi: X\to R^2$ taking (x,y,z) to (x,y)，Find all points of X at which π fails to be a local diffeomorphism

Consider the function $f:R^3\to R$ ， $f(x,y,z)=x^3+xyz+y^2-1$

The set X is the level set： $X=f^{-1}(0)$

f is smooth

$\nabla f=(3x^2+yz, xz+2y, xy)$

Implicit function theorem states that if f is smooth and $\nabla f\neq 0$ for all (x,y,z) $\in X$ ，then X is locally diffeomorphic to R^2 ，this would mean X is a 2-dimensional manifold。

仔細討論 X 上任一點皆不使得 $\nabla f=0$

Then by implicit function theorem， $X=\left\{(x,y,z)\in R^3 \middle| f(x,y,z)=0\right\}$ is a 2-dimensional manifold in R^3 。

X is a **2-manifold** because it is the level set of a smooth function $F:\mathbb{R}^3\to\mathbb{R}$ whose gradient never vanishes on that level set.

Implicit function theorem

DeepSeek 說：

The **Implicit Function Theorem** is a fundamental result in multivariable calculus that provides conditions under which a system of equations can be solved for certain variables in terms of others，even when an explicit solution is not feasible。

Given an equation $F(\mathbf{x},\mathbf{y})=0$, where $\mathbf{x}\in\mathbb{R}^n$ and $\mathbf{y}\in\mathbb{R}^m$, the theorem determines when $\mathbf{y}$ can be expressed as a smooth function $\mathbf{y}=f(\mathbf{x})$ locally near a point $(\mathbf{a},\mathbf{b})$, even if f cannot be written explicitly.

Conditions

1. **Smoothness**: F is continuously differentiable (C^1).
2. **Point Condition**: $F(\mathbf{a},\mathbf{b})=0$.
3. **Non-Degenerate Jacobian**: The Jacobian matrix of F with respect to $\mathbf{y}$, evaluated at $(\mathbf{a},\mathbf{b})$, is invertible. For a single equation $F(x,y)=0$, this reduces to $\frac{\partial F}{\partial y}\neq 0$.

Conclusion

- **Existence**: There exist neighborhoods $U \ni \mathbf{a}$ and $V \ni \mathbf{b}$, and a unique C^1 function $f : U \to V$ such that $F(\mathbf{x}, f(\mathbf{x})) = 0$ for all $\mathbf{x} \in U$.
- **Derivative Formula**: The derivative of f is computable using:

$$Df(\mathbf{x}) = -\left(D_{\mathbf{y}} F(\mathbf{x}, f(\mathbf{x}))\right)^{-1} D_{\mathbf{x}} F(\mathbf{x}, f(\mathbf{x})).$$

For a single equation $F(x, y) = 0$, this simplifies to:

$$\frac{dy}{dx} = -\frac{\partial F / \partial x}{\partial F / \partial y}.$$

Examples

1. **Circle Equation**: $x^2 + y^2 = 1$. At $(0, 1)$, $\frac{\partial F}{\partial y} = 2 \neq 0$, so $y = \sqrt{1 - x^2}$ locally.
2. **Sphere Equation**: $x^2 + y^2 + z^2 = 1$. At $(0, 0, 1)$, $\frac{\partial F}{\partial z} = 2 \neq 0$, so $z = \sqrt{1 - x^2 - y^2}$.

Remarks：

- **Local Nature**: The solution is guaranteed only near the point where the Jacobian is invertible.
- **Sufficient, Not Necessary**: The Jacobian condition is sufficient but not necessary; solutions may exist even if it fails.
- **Relation to Inverse Function Theorem**: A special case where $F(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}) - \mathbf{x}$.

Summary：

The Implicit Function Theorem ensures the local existence，uniqueness，and differentiability of implicitly defined functions under non-degenerate Jacobian conditions，enabling analysis of complex systems without explicit solutions。