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Noether Symmetries Versus Killing Vectors and Isometries of Spacetimes

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Symmetries of spacetime manifolds which are given by Killing vectors are compared with the symmetries of the Lagrangians of the respective spacetimes. We find the point generators of the one parameter Lie groups of transformations that leave invariant the action integral corresponding to the Lagrangian (Noether symmetries). In the examples considered, it is shown that the Noether symmetries obtained by considering the Larangians provide additional symmetries which are not provided by the Killing vectors. It is conjectured that these symmetries would always provide a larger Lie algebra of which the KV symmetres will form a subalgebra.

KEY WORDS: Noether symmetries; isometries of spacetimes; Lie algebras.
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1. INTRODUCTION

The Einstein field equations which govern the general theory of relativity (GR) are described in terms of the 4-Lorentzian metric g_{ab} and are highly non-linear equations. It has therefore been one of the fundamental problems in GR to find and understand solutions of the Einstein field equations through the symmetries they possess, see Meisner *et al.* (1973). These symmetries are given by Killing vectors (KVs): a KV is the one along which the Lie derivative of the metric is zero. Since these symmetries are pivotal to understand the physics of the gravitational fields, they have been throughly investigated and by now a large body of literature is available on them (Petrov, 1969). As fas as the KVs are concerned, they form a finite dimensional Lie group for the spacetime metric being non-degenerate. On the one hand the metric conservation laws are pivotal to study the symmetry groups admitted by them, there are other tensors of more physical interest whose

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symmetries are wroth investigating. These tensors are given by Ricci and Riemann tensors and are fundamentally different from the metric tensor for their being degenerate unlike the metric tensor (Bokhari, 1992). The symmetries of these tensors are generally known as Ricci (RC) and curvature (CC) collineations. It is well known that the RC symmetry forms a larger symmetry group compared with that of the KV or the CC groups even when the RC symmetry is finite. In-fact what is known is that whereas every KV is a CC and every CC is a RC, the converse is not true in general (i.e. KV $\supseteq$ CC $\supseteq$ RC). All these symmetries apart from some other have been widely investigated and studied in comparison with each other. These investigations have not only yielded interesting group theoretic understanding of the symmetries but nice additional physical understanding is also achieved.

With the above point in mind a question arises that "are KVs, RCs and CCs etc the entire symmetries enough to understand GR or could there be other candidates also which might add to some physics not provided by either of the well known symmetres?" It will be of physical interest if such symmetries are investigated and found to form "different" Lie algebras from the "conventional" ones. There exists a nontrivial connection between the structure of the spacetime manifold and the differential equations which are defined by the Euler-Lagrange (geodesic) equations. This system of differential equations are, in turn, connected to various symmetry generators defining transformations of the equations. In particular, the one parameter Lie groups of transformations of the equations are well known and their uses in the analysis of the differential equation like reduction and linearization is well established. Recently, an attempt has been made to establish a connection between differential equations and spacetime manifolds. It has been suggested that this relationship exists via the notions of Lie symmetries of the geodesic equations and isometries of the manifold (Feroze *et al.*).

In this paper, we show that a stronger and more significant connection is defined by the transformations that leave invariant the action integral corresponding to the above Lagrangian and the isometries of the manifold. These Noether symmetries form a subalgebra of the Lie symmetries and double reduction of differential equations (see Olver, 1986; Stephani, 1989; Kara *et al.*, 1994). This implies that these symmetries are closely linked to the geodesic equations and therefore are worth understanding more clearly. More significantly, they have far reaching physical value. For example, each Noether symmetry give rise to a conservation law via the celebrated Noether's theorem (Noether, 1918) (see Stephani, 1989 for some indications but do not contain detailed studies). Of further interest to the Inverse Problem is that given a system of equations representing geodesics, what kind of underlying manifold arises and, also, in the case of non unique Lagrangians with "different" Lie algebra of Noether point symmetry generators, what is the relationship, if any, of the underlying manifolds associated with the given system (see Khesin, 2005).

We note here that as far as second-order scalar (ordinary) differential equations (odes) are concerned, the classifications of the corresponding first-order Lagrangians are complete (Kara, Mahomed, and Leach, 1994).

We briefly state some of the features of an Euler Lagrange system of des. Consider an rth-order system of partial differential equations of n independent and m dependent variables, viz.,

$$E^{\beta}\big(x, u, u_{(1)}, \ldots, u_{(r)}\big) = 0, \quad \beta = 1, \ldots, \tilde{m}. \tag{1.1}$$

A conservation law of (1.1) is the equation

$$D_i T^i = 0, \tag{1.2}$$

on the solutions of (1.1). Here the *total differentiation operator* is

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \ldots, \quad i = 1, \ldots, n.$$

The tuple $T = (T^1, \ldots, T^n)$ is called a *conserved vector* of (1.1).

Suppose $\mathcal{A}$ is the universal space of differential functions. A Lie-Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha} \frac{\partial}{\partial u_{i_1 i_2}^{\alpha}} + \ldots, \tag{1.3}$$

where $\xi^i, \eta^{\alpha} \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned} \zeta_i^{\alpha} &= D_i(W^{\alpha}) + \xi^j u_{ij}^{\alpha}, \\ \zeta_{i_1 i_2}^{\alpha} &= D_{i_1} D_{i_2}(W^{\alpha}) + \xi^j u_{j i_1 i_2}^{\alpha}, \\ &\vdots \end{aligned} \tag{1.4}$$

and W^{α} is the Lie characteristic function defined by

$$W^{\alpha} = \eta^{\alpha} - \xi^j u_j^{\alpha}. \tag{1.5}$$

In this paper, we will assume that X is a Lie point operator, i.e., ξ and η are functions of x and u and are independent of derivatives of u.

The Euler-Lagrange equations, if they exist, associated with (1.1) are the system $\delta L/\delta u^{\alpha} = 0$, $\alpha = 1, \ldots, m$, where $\delta/\delta u^{\alpha}$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^{\alpha}}, \quad \alpha = 1, \ldots, m. \tag{1.6}$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_{\Omega} L\big(x, u, u_{(1)}, \ldots, u_{(r)}\big)\, \mathrm{d}x \tag{1.7}$$

defined over Ω. If we include point dependent gauge terms $f_1, \ldots, f_n$, the Noether symmetries X are given by

$$XL + LD_i\xi_i = D_i f_i. \tag{1.8}$$

Corresponding to each X, a conserved vector $T = (T^1, \ldots, T^n)$ is obtained via Noether's Theorem.

For our purposes, the Lagrangian is obtained directly from the metric which gives rise to the geodesic equations.

2. NOETHER SYMMETRIES/ALGEBRAS OF GEODESIC EQUATIONS

Example 1 (Illustrative)

As an illustrative case, consider the well known Lagrangian

$$L = \dot{r}^2 + r^2\dot{\theta}^2 \tag{2.1}$$

where the dot represents derivative with respect to the arc length parameter "s". In this case, we suppose the form of the vector field X to be $\sigma\frac{\partial}{\partial s} + \phi\frac{\partial}{\partial \theta} + \rho\frac{\partial}{\partial r}$ and substitution in (1.8) leads to, after separation of monomials, the following linear system

$$\begin{aligned}
\dot{\theta}^3 &: \quad -\sigma_\theta r^2 = 0 \\
\dot{r}^3 &: \quad \sigma_r = 0 \\
\dot{\theta}^2 &: \quad 2\rho + 2r\phi_s - r\sigma_s = 0 \\
\dot{r}^2 &: \quad 2\rho_r = \sigma_s \\
\dot{\theta}\dot{r} &: \quad \rho_t = -2r^2\phi_r \\
\dot{\theta} &: \quad 2r^2\phi_s = f_s \\
\dot{r} &: \quad 2\rho_s = f_r \\
1 &: \quad f_s = 0
\end{aligned} \tag{2.2}$$

With some manipulation we get

$$\begin{aligned}
\sigma &= \sigma(s) \\
\rho &= \frac{1}{2}\sigma_s r + \alpha(s, \theta) \\
\phi &= \frac{1}{2r}\alpha_\theta + \underline{\ }(s, \theta)
\end{aligned} \tag{2.3}$$

where, it turns out, $\beta = \beta(s)$, $\alpha_{\theta\theta} + 2\alpha = 0$ so that $\alpha = A(s)\cos\sqrt{2}\theta + B(s)\sin\sqrt{2}\theta$ and $f = 0$. With further analysis we obtain

$$\begin{aligned} \sigma &= b_1 s + b_2 \\ \rho &= \frac{1}{2} r b_1 + k_2 \cos\sqrt{2}\theta + k_4 \sin\sqrt{2}\theta \\ \phi &= \frac{1}{2r}\sqrt{2}[k_2\cos\sqrt{2}\theta - k_4\sin\sqrt{2}\theta] + c_2 \end{aligned} \tag{2.4}$$

We, thus, have five-dimensional algebra of Noether symmetries with basis

$$X_1 = \frac{\partial}{\partial\theta},\ X_2 = \frac{\partial}{\partial s},\ X_3 = s\frac{\partial}{\partial s} + \frac{1}{2} r\frac{\partial}{\partial r}$$

$$X_4 = -\frac{1}{\sqrt{2}r}\sin\sqrt{2}\theta\frac{\partial}{\partial\theta} + \cos\sqrt{2}\theta\frac{\partial}{\partial r},\ X_5 = \frac{1}{\sqrt{2}r}\cos\sqrt{2}\theta\frac{\partial}{\partial\theta} + \sin\sqrt{2}\theta\frac{\partial}{\partial r},$$

and

$$\begin{gathered} [X_1, X_2] = [X_1, X_3] = 0, [X_1, X_4] = -\sqrt{2}X_5, [X_1, X_5] = \sqrt{2}X_4 \\ [X_3, X_2] = -X_2, [X_4, X_2] = [X_5, X_2] = 0, \\ [X_3, X_4] - \tfrac{1}{2}X_4, [X_3, X_5] = -\tfrac{1}{2}X_5, [X_4, X_5] = 0. \end{gathered}$$

Notes.

1. Each of these leads to a conservation law via the celebrated Noether's theorem.
2. As far as the Killing vectors of the two dimensional metric in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{2.5}$$

are concerned, we have

$$\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}, \quad \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial\theta}$$

Comparing, it can be seen that the Killing vectors of the metric form a proper sub set of the Noether symmetries . In-fact the Noether symmetries provides two additional symmetry generators. This inclusion of additional symmetries may be of interest in gaining further insight in situations of physical interest. In fact, X_2 from corresponds to translation in s and X_3 corresponds to dilation and together with the conserved quantity obtained by Noether's theorem is useful, inter alia, in the double reduction of the geodesic (Euler Lagrange) equations.

Example 2

The metric

$$ds^2 = g(r)dt^2 - \frac{1}{g(r)}dr^2 - (r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \tag{2.6}$$

with Lagrangian given by

$$L = g(r)\dot{t}^2 - \frac{1}{g(r)}\dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin^2\theta\dot{\phi}^2 \tag{2.7}$$

where $g = 1 - \frac{2m}{r}$, m a constant, is a four-dimensional Lorentzian metric admitting spherical and static symmetry. It is a non-flat vacuum solution of the Einstein equations (see Bokhari, Kashif and Qadir).

We suppose the Noether point symmetry generator to be of the form

$$X = \sigma\frac{\partial}{\partial s} + \tau\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial r} + J\frac{\partial}{\partial\theta} + F\frac{\partial}{\partial\phi}.$$

Substituting in (1.8) leads to

$$\begin{aligned}
&\rho\left[\frac{2m}{r^2}\dot{t}^2 + \frac{2m}{(r-2m)^2}\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\dot{\phi}^2\right] + J[-2r^2\sin\theta\cos\theta\dot{\phi}^2] \\
&\quad + 2\left(1 - \frac{2m}{r}\right)\dot{t}(\tau_s + \dot{t}\tau_t + \dot{r}\tau_r + \dot{\theta}\tau_\theta + \dot{\phi}\tau_\phi) \\
&\quad - 2\dot{t}^2\left(1 - \frac{2m}{r}\right)(\sigma_s + \dot{t}\sigma_t + \dot{r}\sigma_r + \dot{\theta}\sigma_\theta + \dot{\phi}\sigma_\phi) \\
&\quad - \frac{2r}{r-2m}\dot{r}(\rho_s + \dot{t}\rho_t + \dot{r}\rho_r + \dot{\theta}\rho_\theta + \dot{\phi}\rho_\phi) \\
&\quad + \frac{2r}{2-2m}\dot{r}^2(\sigma_s + \dot{t}\sigma_t + \dot{r}\sigma_r + \dot{\theta}\sigma_\theta + \dot{\phi}\sigma_\phi) \\
&\quad - 2r^2\dot{\theta}(J_s + \dot{t}J_t + \dot{r}J_r + \dot{\theta}J_\theta + \dot{\phi}J_\phi) \\
&\quad + 2r^2\theta^2(\sigma_s + \dot{t}\sigma_t + \dot{r}\sigma_r + \dot{\theta}\sigma_\theta + \dot{\phi}\sigma_\phi) \\
&\quad - 2r^2\sin^2\theta\dot{\phi}((F_s + \dot{t}F_t + \dot{r}F_r + \dot{\theta}F_\theta + \dot{\phi}F_\phi) \\
&\quad + 2r^2\sin^2\theta\dot{\phi}^2(\sigma_s + \dot{t}\sigma_t + \dot{r}\sigma_r + \dot{\theta}\sigma_\theta + \dot{\phi}\sigma_\phi) \\
&\quad + \left(1 - \frac{2m}{r}\right)\dot{t}^2(\sigma_s + \dot{t}\sigma_t + \dot{r}\sigma_r + \dot{\theta}\sigma_\theta + \dot{\phi}\sigma_\phi) \\
&\quad - \frac{r}{r-2m}\dot{r}^2(\sigma_s + \dot{t}\sigma_t + \dot{r}\sigma_r + \dot{\theta}\sigma_\theta + \dot{\phi}\sigma_\phi)
\end{aligned}$$

$$
\begin{aligned}
&- r^2\dot{\theta}^2(\sigma_s + \dot{t}\sigma_t + \dot{r}\sigma_r + \dot{\theta}\sigma_\theta + \dot{\phi}\sigma_\phi)\\
&- r^2\sin^2\theta\dot{\phi}^2(\sigma_s + \dot{t}\sigma_t + \dot{r}\sigma_r + \dot{\theta}\sigma_\theta + \dot{\phi}\sigma_\phi)\\
= {} & f_s + \dot{t} f_t + \dot{r} f_r + \dot{\theta} f_\theta + \dot{\phi} f_\phi.
\end{aligned} \tag{2.8}
$$

The separation by monomials lead to $f_s = 0$ and

$$
\begin{aligned}
\sigma &= \sigma(s),\\
\rho\frac{2m}{r^2} + 2\left(1 - \frac{2m}{r}\right)\tau_t - \left(1 - \frac{2m}{r}\right)\sigma_s &= 0,\\
\rho\frac{2m}{(r-2m)^2} - \rho_r\frac{2r}{(r-2m)} + \frac{r}{(r-2m)}\sigma_s &= 0,\\
-2\rho r - 2r^2 J_\theta + r^2\sigma_s &= 0,\\
\rho(-2r\sin^2(\theta) - 2Jr^2\sin(\theta)\cos(\theta) - 2r^2\sin^2(\theta)F_\phi + r^2\sin^2(\theta)\sigma_s &= 0,\\
2\left(1 - \frac{2m}{r}\right)\tau_r - \frac{2r}{r-2m}\rho_t &= 0,\\
2\left(1 - \frac{2m}{r}\right)\tau_\theta - 2r^2 J_t &= 0,\\
2\left(1 - \frac{2m}{r}\right)\tau_\phi - 2r^2\sin^2(\theta)F_t &= 0,\\
-\frac{2r}{r-2m}\rho_\theta - 2r^2 J_r &= 0,\\
-\frac{2r}{r-2m}\rho_\phi - 2r^2\sin^2(\theta)F_r &= 0,\\
-2r^2 J_\phi - 2r^2\sin^2(\theta)F_\theta &= 0,\\
f_s &= 0,\\
2\left(1 - \frac{2m}{r}\right)\tau_s &= f_t,\\
-\frac{2r}{r-2m}\rho_s &= f_r,\\
-2r^2 J_s &= f_\theta,\\
-2r^2\sin^2(\theta)F_s &= f_\phi
\end{aligned} \tag{2.9}
$$

It is clear that solving this system can be tedious. However, we list a subalgebra that arises, viz.,

$$X_1 = \frac{\partial}{\partial s}, X_2 = \frac{\partial}{\partial t}, X_3 = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi},$$
$$X_4 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, X_5 = \frac{\partial}{\partial\phi} \tag{2.10}$$

The nonzero commutators are

$$[X_3, X_4] = -X_5, [X_4, X_5] = -X_3, [X_3, X_5] = X_4 \tag{2.11}$$

Example 3

We now attempt the Noether symmetries of the Lagrangian for a 4-dimensional Lorentzian (Minkowski) metric. This is a spacetime admitting maximal KV and arbitrary RC and CC symmetry: arbitrary RC and CC symmetry because the spacetime is flat and all the Ricci and Riemann tensor components for it are zero. The spacetime metric has the form

$$ds^2 = \cosh\left(\frac{x}{a}\right)dt^2 - dx^2 - (dy^2 + dz^2) \tag{2.12}$$

To determine its Noether symmetries we use the Lagrangian given by

$$L = \cosh\left(\frac{x}{a}\right)\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 \tag{2.13}$$

in (1.8) with point symmetry generator $\mathcal{X} = \sigma\frac{\partial}{\partial s} + T\frac{\partial}{\partial t} + Y\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}$ and $f = f(s, t, x, y, z)$, i.e.,

$$\mathcal{X}^{[1]}\left(\cosh\left(\frac{x}{a}\right)\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2\right) + \left(\cosh\left(\frac{x}{a}\right)\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2\right)D_s\sigma = D_{sf}. \tag{2.14}$$

Separating by the respective monomials, we get the following nontrivial system of linear pdes

$$\dot{t}^3: \quad -\cosh^2\left(\frac{x}{a}\right)\sigma_t = 0$$

$$\dot{x}^3: \quad -\sigma_x = 0$$

$$\dot{y}^3: \quad -\sigma_y = 0$$

$$\dot{z}^3: \quad -\sigma_z = 0$$

$$\dot{t}^2: \quad \frac{2}{a}\cosh\left(\frac{x}{a}\right)\sinh\left(\frac{x}{a}\right)X + 2\cosh^2\left(\frac{x}{a}\right)T_t - \cosh^2\left(\frac{x}{a}\right)\sigma_s = 0$$

$$\dot{x}^2: \quad 2X_x - \sigma_s = 0$$

$$
\begin{aligned}
\dot{y}^2:&\quad 2Y_y - \sigma_s = 0\\
\dot{z}^2:&\quad 2\zeta_z - \sigma_s = 0\\
\dot{t}\dot{x}:&\quad X_t = \cosh^2\left(\frac{x}{a}\right)T_x\\
\dot{t}\dot{y}:&\quad Y_t = \cosh^2\left(\frac{x}{a}\right)T_y\\
\dot{t}\dot{z}:&\quad \zeta_t = \cosh^2\left(\frac{x}{a}\right)T_z\\
\dot{y}\dot{x}:&\quad -X_y = Y_x\\
\dot{z}\dot{x}:&\quad -X_z = \zeta_x\\
\dot{y}\dot{z}:&\quad -Y_z = \zeta_y\\
\dot{t}:&\quad 2\cosh^2\left(\frac{x}{a}\right)T_s = f_t\\
\dot{x}:&\quad -2X_s = f_x\\
\dot{y}:&\quad -2Y_s = f_y\\
\dot{z}:&\quad -2\zeta_s = f_z\\
1:&\quad f_s = 0
\end{aligned}
\tag{2.15}
$$

Concentrating on the forms of the Noether symmetries $\mathcal{X}$ we get after some tedious calculations (the forms of the corresponding gauge functions f can be determined from the last five equations in (2.16) which are needed if one requires the conserved quantities via Noether's theorem),

$$
\begin{aligned}
T &= \tanh\left(\frac{x}{a}\right)\left[e_0\cos\left(\frac{t}{a}\right) - e_2\sin\left(\frac{t}{a}\right) + D^1(y,z)\right]\\
X &= e_2\cos\left(\frac{t}{a}\right) + e_0\sin\left(\frac{t}{a}\right) + e_5\\
Y &= b^1(s,y,z)t + b^2(s,y,z)\\
\zeta &= c^1(s,y,z)t + c^2(s,y,z)\\
\sigma &= e_1
\end{aligned}
\tag{2.16}
$$

where the e_i's are constants and

$$
D^1_y = b^1, \qquad D^1_z = c^1, \qquad c^1_y = -b^1_z, \qquad c^2_y = -b^2_z.
$$

Thus, we get

$$\begin{aligned}
\mathcal{X}_1 &= \tanh\left(\frac{x}{a}\right)\cos\left(\frac{t}{a}\right)\frac{\partial}{\partial t} + \sin\left(\frac{t}{a}\right)\frac{\partial}{\partial x}\\
\mathcal{X}_2 &= -\tanh\left(\frac{x}{a}\right)\sin\left(\frac{t}{a}\right)\frac{\partial}{\partial t} + \cos\left(\frac{t}{a}\right)\frac{\partial}{\partial x}\\
\mathcal{X}_3 &= \frac{\partial}{\partial x}\\
\mathcal{X}_4 &= \frac{\partial}{\partial s}\\
\mathcal{X}_\infty &= D^1\frac{\partial}{\partial t} + (b^1 t + b^2)\frac{\partial}{\partial y} + (C^1 t + c^2)\frac{\partial}{\partial z}
\end{aligned}$$

Without further analysis, we are in a position to list some of the Noether symmetries as a consequence of $\mathcal{X}_\infty$, for e.g.,

$$\begin{aligned}
D^1 = 1: &\quad \frac{\partial}{\partial t}\\
D^1 = y: &\quad y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}\\
D^1 = z: &\quad z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z}\\
b^2 = y, c^2 = z: &\quad y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\\
b^2 = z, c^2 = -y: &\quad z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\\
b^2 = 1: &\quad \frac{\partial}{\partial y}y\\
c^2 = 1: &\quad \frac{\partial}{\partial z}
\end{aligned} \tag{2.17}$$

where, respectively, these are time translation, Lorentz rotation in y, Lorentz rotation in z, dilation in $y - z$, rotation in $y - z$, translation in y and translation in z.

3. CONCLUSIONS

We have shown that the Killing vectors of the metric form a subalgebra of the Noether symmetries arising from the "usual" Lagrangian. This has a number of implications on finding solutions to some unsolved problems. For example, the study of the symmetry relationship between differential equations and the underlying manifolds has only recently begun; the algebra of symmetries certainly has bearing on this study. Secondly, whilst it is well known that the algebra of Noether symmetries form a subalgebra of Lie point symmetries of a differential equations, there always existed the question of the non uniqueness of Lagrangians of differential equations. That is, what is the relationship between the "unification" of the various Noether algebras that arise and the Lie algebra of point symmetries of the Euler-Lagrange equations?

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REFERENCES

Bokhari, A. H. (1992). *International Journal of Theoretical Physics* **31**, 2091.

Bokhari, A. H., Kashif, A. R., and Qadir, A. (2004). Curvature collineations of some plane symmetric static spacetimes, to appear in *Proceedings of the 11th Regional Conference, Teheran, Iran*.

Feroze, Tooba, Mahomed, F. M., and Qadir, Asghar (2005). The Connection Between Isometries and Symmetries of the Underlying Spaces, to appear in *Nonlinear Dynamics*.

Kara, A. H., Mahomed, F. M., and Leach, P. G. L. (1994). The Noether equivalence problem for particle Lagrangians. *J. Math. Anal. Appl.* **188**(3), 867–884.

Kara, A. H., Vawda, F. E., and Mahomed, F. M. (1994). Symmetries of first integrals and solutions of differential equations. *Lie Groups and their Applications* **1**(2), 1–17.

Khesin, B. (2005). Topological fluid dynamics. *Notices of the AMS* **52**(1), 9–19.

Meisner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation* Benjamin, New York.

Noether, E., Invariante Variationsprobleme (1918). *Nachrichten der Akademie der Wissenschaften in Göttingen, Mathematisch-Physikalische Klasse*, **2**, 235–257 (English translation in *Transport Theory and Statistical Physics*, **1**(3), 186–207 (1971)).

Olver, P. (1986). *Application of Lie Groups to Differential Equations* Springer-Verlag, New York.

Petrov, A. Z. (1969). *Einstein Spaces*, Pergamon, Oxford.

Stephani, H. (1989). *Differential Equations: Their Solution Using Symmetries*. Cambridge University Press, Cambridge.