REDSHIFTS and KILLING VECTORS

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Abstract

Courses in introductory special and general relativity have increasingly become part of the curriculum for upper-level undergraduate physics majors and master's degree candidates. One of the topics rarely discussed is symmetry, particularly in the theory of general relativity. The principal tool for its study is the Killing vector. We provide an elementary introduction to the concept of a Killing vector field, its properties, and as an example of its utility apply these ideas to the rigorous determination of gravitational and cosmological redshifts.

1 Prologue

In 1907, Albert Einstein enunciated his "equivalence principle" and used it to examine the influence of the gravitational field on the propagation of light[1]. He demonstrated, in approximation that as light moved through a difference in gravitational potential its frequency would change. It is with the latter that the concept of *gravitational redshift* enters the ken of the physicist.

In 1916 Einstein completed his journey to a theory of gravitation: *The Foundations of the General Theory of Relativity*[2]. Here, redshift is studied with greater precision. It is discussed in terms of the *metric*. In Section 22, *Behavior of Rods and Clocks in the Static Gravitational Field*, toward the end of the paper he states, (in translation) "From this it follows that the spectral lines of light reaching us from the surface of large stars must appear displaced toward the red end of the spectrum."

The General Theory of Relativity describes gravitation as the curvature of spacetime. The Einstein equations are a protocol for the determination, subject to hypotheses concerning the physical nature of the spacetime being discussed, of the metric of the space in question. This metric is the source of *all* information about the properties of the space. In particular, of special interest here, it enables the calculation of the paths of light rays—the null geodesics—and the description of the world lines of objects in the spacetime.

These ideas were substantially refined and put into the form in use today by the distinguished mathematician Hermann Weyl. He introduced the concept of world lines of emitter and observer and the connection of elements of proper time on each defined by null geodesics propagating from the former to the latter. This, for the first time, provided a general definition of frequency shift for light in arbitrary spacetimes. It was first stated in the fifth edition of his book Space-Time-Matter[3]. In Appendix III "Redshift and Cosmology" he explains:

> "The different points on the world line of a point-like light source are the origin of (three-dimensional) surfaces of constant phase that form null cones opening towards the future. From the rhythm of the change of phase on this line one obtains the perceived change of phase for any observer by checking how his world line intersects the successive surfaces of constant phase. Let s be the proper time of the light source and s' the proper time of the observer. To each point s on the world line of the light source corresponds a point s' on the world line of the observer: $s' = s'(s)$, the intersection of his world line with the future null cone emanating from s. If the process occurring at the position of the source is

a purely periodic one of infinitely small period then the change of phase experienced by the observer is also periodic; but the period is increased in the ratio $1 + z = ds'/ds$ (obviously measured along both world lines in their respective proper times). If the observer carries a light source of the same physical nature as the observed one then every spectral line of his light source with frequency f corresponds to the spectral line of the distant light source with frequency $f/(1+z)$. The ones appear to be displaced with respect to the others."

In the ray approximation, light source and observer are connected by null geodesics running on surfaces of constant phase of the future light cones. (See Fig. 1). Null geodesics emitted at proper times s and $s + ds$ will intersect the world line of the observer at proper times s' and $s' + ds'$ respectively. (See Fig. 2). The redshift is simply

$$1 + z = \frac{ds'}{ds} \ . \tag{1.1}$$

In order to find this ratio one must know the null geodesics, or more precisely, their variation. In general this is not simple. The task is greatly simplified if, under a time translation, the geodesic is not changed or is subjected only to a scale transformation. *These are the conditions, respectively, for the existence of either a Killing vector or a conformal Killing vector in the spacetime manifold.* This is the central point of our investigation.

2 Introduction

The concept of symmetry is central to the solution of many problems in physics. Introduction of ignorable coordinates in the construction of kernel functions for Lagrangians entails implicit use of *a priori* knowledge of the symmetries of the system being studied. Construction of the Hamiltonian for a problem in quantum mechanics is constrained by the symmetries thereof. It is the measure of the power of symmetry considerations that solely through their application, the Robertson-Walker metric can be constructed. In this paper we study redshift and, as we shall show, the symmetries of the spacetime determine the manner in which redshifts occur.

Spacetime is a 4-dimensional Riemannian manifold, that is, it is a surface. The symmetries of a surface are numerous: among these there are discrete symmetries such as inversion in a point or reflections in a plane and there are the infinitesimal continuous coordinate transformations (also called mappings or motions) which *leave* *the metric unchanged*. Such mappings are called "isometries". It is the latter which will occupy our attention.

A significant, yet obvious example is furnished by the isometries of the metric of special relativity. Its "flatness" enables their simple enumeration: 3 spatial translations, 3 rotations, 3 pure Lorentz transformations, and the translations along the time axis. This last is worth noting. In a 4-dimensional space, time is the same as any other coordinate. If a metric is invariant under translations in the x-direction this is no different in principle than saying that a metric is invariant under translations along the t-axis. The latter implies that the metric is stationary. Conversely, if a metric is stationary it possesses a time symmetry.

Other well-known examples are the sphere which has the symmetry of the well-known three-dimensional rotation group, $\mathbf{O}_3$. In the curved spacetime of general relativity things are rarely that obvious. The continuous symmetries are those of interest here.

The principal tool for investigating the isometries of a metric is the Killing vector field[4,5] which was introduced by the late 19th century German mathematician Wilhelm Killing, its distinguished eponym. It was developed and exploited in the study of continuous groups. The set of Killing vectors for a given metric provides an invariant characterization of these properties. No matter the coordinate system in which the metric is cast, its set of Killing vectors (*modulo* coordinate transformations) will be the same.

3 Killing Vector Fields

Consider the infinitesimal change in a metric, g_{ab}, generated by a vector field, $\mathbf{f}$:

$$\tilde{x}^\alpha = x^\alpha + \epsilon f^\alpha(x^\beta)\,, \tag{3.1}$$

where ϵ is an infinitesimal constant[e]. The result of the mapping Eq. (3.1) (see Fig. 3) is to move a point $P(x^\alpha)$ to point P' with coordinates $x^\alpha + \epsilon f^\alpha(x^\beta)$. Similarly, a neighboring point $Q(x^\alpha + dx^\alpha)$ will be moved to point $Q'(x^\alpha + dx^\alpha + \epsilon f(x^\alpha + dx^\alpha))$ (or, up to first order in differentials, $Q'(x^\alpha + dx^\alpha + \epsilon f^\alpha + \epsilon f^\alpha{}_{,\gamma} dx^\gamma)$.

The infinitesimal interval $\overline{P'Q'}$ is

$$(x^\alpha + dx^\alpha + \epsilon f^\alpha + \epsilon f^\alpha{}_{,\gamma} dx^\gamma) - (x^\alpha + \epsilon f^\alpha) \tag{3.2}$$

or

$$dx^\alpha + \epsilon f^\alpha{}_{,\gamma} dx^\gamma\,. \tag{3.3}$$

The length, $\overline{ds}^2$, of this interval is given by

$$\begin{aligned}\overline{ds}^2 &= g_{\alpha\beta}(x^\gamma + \epsilon f^\gamma)(dx^\alpha + \epsilon f^\alpha{}_{,\kappa} dx^\kappa)(dx^\beta + \epsilon f^\beta{}_{,\kappa} dx^\kappa) \\ &= (g_{\alpha\beta} + \epsilon g_{\alpha\beta,\gamma} f^\gamma)(dx^\alpha + \epsilon f^\alpha{}_{,\gamma} dx^\gamma)(dx^\beta + \epsilon f^\beta{}_{,\kappa} dx^\kappa)\,. \end{aligned} \tag{3.4}$$

Expanding Eq. (3.4), neglecting terms of order ϵ^2, and rearranging dummy indices results in

$$\overline{ds}^2 = g_{\alpha\beta} dx^\alpha dx^\beta + \epsilon(g_{\alpha\beta,\gamma} f^\gamma + g_{\alpha\gamma} f^\gamma_{,\beta} + g_{\gamma\beta} f^\gamma{}_{,\alpha}) dx^\alpha dx^\beta\,. \tag{3.5}$$

With the definition

$$2s_{\alpha\beta} \equiv g_{\alpha\beta,\gamma} f^\gamma + g_{\alpha\gamma} f^\gamma{}_{,\beta} + g_{\gamma\beta} f^\gamma{}_{,\alpha} \tag{3.6}$$

the change in the metric may be written as

$$\frac{1}{\epsilon}(\overline{ds}^2 - ds^2) = 2s_{\alpha\beta} dx^\alpha\, dx^\beta\,. \tag{3.7}$$

Because the left hand side of this equation is a scalar and $dx^\alpha\, dx^\beta$ is a symmetric tensor it may be inferred that the symmetric quantity $s_{\alpha\beta}$ is a covariant tensor[6].

The structure of the right hand side of Eq. (3.6)

$$g_{\alpha\beta,\gamma} f^\gamma + g_{\alpha\gamma} f^\gamma{}_{,\beta} + g_{\gamma\beta} f^\gamma{}_{,\alpha} \tag{3.8}$$

combines partial differentiation and a vector field and arises in precisely this form in many applications. The operation is important enough to have its own name and symbol. It is called the "Lie derivative" of a geometric object (in this instance, the metric tensor, g_{ab}) with respect to a vector field, f, and is written $\mathcal{L}_f$. Thus, Eq. (3.6) may be written as

$$\mathcal{L}_f g_{\alpha\beta} = 2s_{\alpha\beta}\,. \tag{3.9}$$

In the special case where the metric tensor is invariant under the transformation we will have $\mathcal{L}_f g_{\alpha\beta} = 0$. Then the vector $\mathbf{f}$ is, by definition, a *Killing vector*. Killing vectors are customarily designated by the symbol $\boldsymbol{\xi}$. For Killing vectors, then, we have

$$\mathcal{L}_\xi g_{\alpha\beta} = g_{\alpha\beta,\gamma}\xi^\gamma + g_{\alpha\gamma}\xi^\gamma{}_{,\beta} + g_{\beta\gamma}\xi^\gamma{}_{,\alpha} = 0\,. \tag{3.10}$$

This equation can be written in a different form, useful for many purposes, *viz.*,

$$\mathcal{L}_\xi g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0\,. \tag{3.11}$$

The identity of these 2 forms can be demonstrated by expressing both in geodetic coordinates. The result is identical expressions. Consequently, they are identical in any choice of coordinates. Alternatively, expansion of the covariant derivatives of Eq. (3.11) yields precisely Eq. (3.10). Any 2-index tensor may be expressed as the sum of a symmetric and skew-symmetric tensor. Equation (3.11) indicates that the symmetric part of the tensor $\xi_{\alpha;\beta}$ vanishes. Because infinitesimal displacements generated by Killing vectors leave the metric unchanged, these displacements map geodesics onto neighboring geodesics. Note, also, that Eqs. (3.6) may be read in either of two ways. Given a metric, they provide the means for the determination of its Killing vectors or given a set of Killing vectors, determining the metric. The latter is ill-defined.

4 Killing Vectors in Flat Spacetimes

In flat spacetimes Cartesian coordinates may be introduced. Consequently, in Eqs. (3.11), covariant derivatives may be replaced by partial derivatives and the right hand portion becomes

$$\xi_{\alpha,\beta} + \xi_{\beta,\alpha} = 0\,. \tag{4.1}$$

By differentiation we get

$$\xi_{\alpha,\beta,\gamma} + \xi_{\beta,\alpha,\gamma} = 0 \tag{4.2}$$

and by cyclic permutation of the indices we obtain

$$\xi_{\beta,\gamma,\alpha} + \xi_{\gamma,\beta,\alpha} = 0 \tag{4.3}$$

and

$$\xi_{\gamma,\alpha,\beta} + \xi_{\alpha,\gamma,\beta} = 0\,. \tag{4.4}$$

Addition of Eqs. (4.2) and (4.3), subtraction of Eq. (4.4), and recognition that second partial derivatives commute, results in the differential equations

$$2\xi_{\gamma,\alpha,\beta} = 0\,. \tag{4.5}$$

The solutions of these equations are the general linear functions

$$\xi_\alpha = A_{\alpha\beta} x^\beta + B_\alpha \tag{4.6}$$

where $A_{\alpha\beta}$ and B_α are constants. If Eq. (4.6) is substituted into Eq. (4.1) we find immediately that $A_{\alpha\beta}$ is skew-symmetric

$$A_{\alpha\beta} = -A_{\beta\alpha}\,. \tag{4.7}$$

We see, thus, that an n-dimensional flat space has $n(n+1)/2$ independent Killing vectors. For Minkowski spacetime this is just 10. This demonstration depends in an essential way on the *flatness* of the spacetime. It is this fact which permits the substitution of partial for covariant differentiation and consequent ability to reorder the differentiation.

5 Conformal and Homothetic Motions

In addition to the mappings described by Killing vectors, there are other classes of transformations generated by vector fields which are important in the present context. These are the so called "homothetic" and "conformal" motions. In these cases, respectively, the metric tensor is either multiplied by a constant or a scalar function. The generators are termed homothetic or conformal Killing vectors. In these cases the stress tensor is *proportional* to the metric. Both cases are subsumed in

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = g_{\alpha\beta,\gamma}\xi^\gamma + g_{\alpha\gamma}\xi^\gamma{}_{,\beta} + g_{\gamma\beta}\xi^\gamma{}_{,\alpha} = 2\sigma g_{\alpha\beta}\ . \tag{5.1}$$

It is simple to determine σ. Contract Eq. (5.1) with $g^{\alpha\beta}$; simple manipulation yields $\sigma = \xi^\gamma{}_{;\gamma}/n$ and

$$g_{\alpha\beta,\gamma}\xi^\gamma + g_{\alpha\gamma}\xi^\gamma{}_{,\beta} + g_{\gamma\beta}\xi^\gamma{}_{,\alpha} = \frac{2}{n}\xi^\gamma{}_{;\gamma}g_{\alpha\beta} \tag{5.2}$$

where n is the dimension of the manifold. If $\xi^\gamma{}_{;\gamma}$ is a constant, the Killing vector, by definition, describes homothetic motions; and if $\xi^\gamma{}_{;\gamma}$ is a scalar field, say $\phi(x^\alpha)$, the motion is called conformal.

Similarly to Eqs. (3.11) this may be written as

$$\begin{aligned} \xi_{\alpha;\beta} + \xi_{\beta;\alpha} &= \frac{2}{n}\xi^\gamma{}_{;\gamma}g_{\alpha\beta} \quad \text{or} \\ \mathcal{L}_\xi g_{\alpha\beta} &= \frac{2}{n}\xi^\gamma{}_{;\gamma}g_{\alpha\beta} \end{aligned} \tag{5.3}$$

In this instance the trace-free symmetric part of $\xi_{\alpha;\beta}$ vanishes.

Again, for flat spacetimes, the situation is vastly simplified. A special homothetic Killing vector is given by

$$\xi^\alpha = \kappa x^\alpha \tag{5.4}$$

where κ is a constant. (See Fig. 4.) The most general homothetic Killing vector is

$$\xi_\alpha = (\kappa\eta_{\alpha\beta} + A_{\alpha\beta})x^\beta + B_\alpha \tag{5.5}$$

where $A_{\alpha\beta} = -A_{\beta\alpha}$. This is readily verified by substitution into in Eq. (5.1) and replacing covariant by partial derivatives. In this instance the coordinate grid is uniformly stretched or shrunk. Similarly, it is easy to confirm that

$$\xi_\alpha = (\eta_{\alpha\beta}C_\gamma - \frac{1}{2}\eta_{\beta\gamma}C_\alpha)x^\beta x^\gamma \tag{5.6}$$

where $\eta_{\alpha\beta}$ is the flat space metric and C_γ are constants, are special conformal Killing vectors.

6 Redshifts Derived from Killing Vectors

The calculation of redshifts is extremely simple in spacetimes possessed of time-like conformal Killing vector fields parallel to the world lines of source and observer. Consider Eq. (1.1), Weyl's universal definition of redshift,

$$1 + z = \frac{ds'}{ds}\,. \tag{6.1}$$

The event $P(x^\alpha)$ at the source is connected to the event $Q(y^\alpha)$ at the observer by a null geodesic. (See Fig 4.) A conformal Killing vector field ξ^a moves $P(x^\alpha)$ into $P'(x^\alpha + \epsilon\xi^a(x^\beta))$and $Q(y^\alpha)$ into $Q'(y^\alpha + \epsilon\xi^a(y^\beta))$ and the null geodesic connecting P and Q into the null geodesic connecting P' and Q'. We have then respectively

$$ds = \mid \epsilon\xi^a(x^\beta) \mid \quad \text{and} \quad ds' = \mid \epsilon\xi^\alpha(y^\beta) \mid \tag{6.2}$$

and thus

$$1 + z = \sqrt{\frac{g_{\alpha\beta}(y^\gamma)\xi^\alpha(y^\gamma)\xi^\beta(y^\gamma)}{g_{\alpha\beta}(x^\gamma)\xi^\alpha(x^\gamma)\xi^\beta(x^\gamma)}}\,. \tag{6.3}$$

This holds if the Killing vector field is tangent to the world line of the source at its point of emission and tangent to the observer's world line at the point of reception. If source and observer move on conformal time-like Killing lines this condition is fulfilled at all times.

7 Doppler Effect in Minkowski Spacetime

Weyl's definition of redshift, $1 + z = ds'/ds$, readily provides the usual formula for relativistic Doppler effect in flat spacetime. Take the world line of the source to be

$$\begin{aligned} x^0 &= t \\ x &= y = z = 0\,. \end{aligned} \tag{7.1}$$

For the world line of the (inertial) observer we take a straight line through the origin with slope β

$$x = \beta t, \quad y = z = 0\,. \tag{7.2}$$

Light emitted by the source at time t will be received by the observer at time t'. (See Fig. 5.) If the observer is receding from the source he or she will see the light at

$$x' = \beta t' \quad y' = z' = 0\,. \tag{7.3}$$

The homothetic Killing vector (5.4) has the lengths, respectively, at source and observer of κt and $\kappa(t'^2 - x'^2)^{1/2}$. We readily obtain from Eq. (6.3)

$$\begin{aligned} 1+z &= \sqrt{\frac{g_{\alpha\beta}(y^\gamma)\xi^\alpha(y^\gamma)\xi^\beta(y^\gamma)}{g_{\alpha\beta}(x^\gamma)\xi^\alpha(x^\gamma)\xi^\beta(x^\gamma)}} \\ &= \frac{\sqrt{t'^2 - x'^2}}{t} \\ &= \frac{t'\sqrt{1-\beta^2}}{t' - x'} \\ &= \frac{\sqrt{1-\beta^2}}{1-\beta} \\ &= \sqrt{\frac{1+\beta}{1-\beta}} \end{aligned} \tag{7.4}$$

This is the relativistic Doppler formula for a receding observer. For an approaching observer replace β by $-\beta$.

It is instructive to apply Weyl's definition of redshift, Eq. (1.1) to Minkowski spacetime with the spatial part expressed in polar spherical coordinates.

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{7.5}$$

We introduce *4-dimensional* polar coordinates with the coordinate transformation

$$\begin{pmatrix} t \\ r \end{pmatrix} = \begin{pmatrix} T\cosh\chi \\ T\sinh\chi \end{pmatrix}\,. \tag{7.6}$$

The differentials are

$$\begin{aligned} dt &= dT\cosh\chi + T\sinh\chi d\chi \\ dr &= dT\sinh\chi + T\cosh\chi d\chi \end{aligned} \tag{7.7}$$

The result is

$$ds^2 = dT^2 - T^2\left[d\chi^2 + \sinh^2\chi\ (d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{7.8}$$

This maps the interiors of the past and future light-cones of Minkowski spacetime into, respectively, linearly contracting and expanding spaces of constant negative curvature. (See Fig. 6.) A straight world line through the origin would be just

$$\beta = \frac{r}{t} \tag{7.9}$$

which by virtue of Eqs. (7.6) would map into

$$\beta = \tanh\chi\,. \tag{7.10}$$

In the mapped space, such lines are, at constant $\{\chi,\, \theta,\, \phi\}$ with χ as the *rapidity*. We now use the homothetic Killing vector Eq. (5.4) to construct a linear first-order differential form

$$\xi_\alpha dx^\alpha = \kappa x_\alpha dx^\alpha = \kappa(t\,dt - r\,dr)\,. \tag{7.11}$$

In Eq. (7.8), with use of Eqs.(7.7), this is readily found to be

$$\xi_\alpha dx^\alpha = \kappa T dT\,. \tag{7.12}$$

The length of the homothetic killing vector in the new coordinate system is thus κT and the redshift is

$$1 + z = \frac{T'}{T}\,. \tag{7.13}$$

8 More Redshifts

8.1 Cosmological Redshifts

The last example leads us directly to the calculation of redshift in the Friedmann cosmological models[7,8]

$$ds^2 = dt^2 - R^2(t)[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{8.1}$$

where S determines the curvature of the 3-space,

$$S = \left\{ \begin{array}{rl} \sin\chi\,, & \text{positive} \\ \chi\,, & \text{flat} \\ \sinh\chi\,, & \text{negative} \end{array} \right\}. \tag{8.2}$$

With Eq. (5.2)

$$g_{\alpha\beta,\gamma}\xi^\gamma + g_{\alpha\gamma}\xi^\gamma{}_{,\beta} + g_{\gamma\beta}\xi^\gamma{}_{,\alpha} = \frac{2}{n}\xi^\gamma{}_{;\gamma} g_{\alpha\beta} \tag{8.3}$$

it is readily confirmed that

$$\xi^\alpha = R(t)\delta^\alpha{}_0 \tag{8.4}$$

is a conformal Killing vector tangent to the world line of the source at $\chi = 0$ and also tangent to the observer's world line at $\{\chi,\, \theta,\, \phi = \mathit{constant}\}$.

It follows from application of Eq. (6.3) that the redshift is

$$1 + z = \frac{R(t')}{R(t)} \tag{8.5}$$

for a light signal emitted at time t and received at t'.

8.2 Redshifts in Stationary Spacetimes

The general stationary spacetime may be written as

$$ds^2 = g_{oo}dt^2 + 2g_{oi}dtdx^i + g_{ij}dx^i dx^j \tag{8.6}$$

with $\partial g_{\alpha\beta}/\partial t = 0$.

Because the field is stationary it necessarily possesses a time-like Killing vector. The simplest assumption is

$$\xi^\alpha = \delta^\alpha{}_0 \, . \tag{8.7}$$

This is readily verified by use of Eq. (3.10)

$$g_{\alpha\beta,\gamma}\xi^\gamma + g_{\alpha\gamma}\xi^\gamma{}_{,\beta} + g_{\beta\gamma}\xi^\gamma{}_{,\alpha} = 0 \, . \tag{8.8}$$

We then have, as earlier,

$$\begin{aligned} 1+z &= \sqrt{\frac{g_{\alpha\beta}(x')\xi^\alpha(x')\xi^\beta(x')}{g_{\gamma\delta}(x)\xi^\gamma(x)\xi^\delta(x)}} \\ &= \sqrt{\frac{g_{oo}(x')}{g_{oo}(x)}} \, . \end{aligned} \tag{8.9}$$

If the world line of the source is $x^j = const$ and that of the observer is $x'^j = const$, both are tangent to a Killing vector of this field.

One of the most important of the stationary metrics is the Schwarzschild-Droste (SD) metric It is scarcely known among relativists that the determination of the metric for a point mass was accomplished almost simultaneously with Karl Schwarzschild by Johannes Droste, a Ph.D. student of H. A. Lorentz. The form of the metric (8.10) was actually due to Droste[9].

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \, . \tag{8.10}$$

By virtue of the Birkhoff theorem[10,11], all spherically symmetric, vacuum metrics are stationary and equivalent to Eq. (8.10). Application of Eq. (8.9) yields the redshift

$$1+z = \sqrt{\frac{1-2m/r'}{1-2m/r}} \, . \tag{8.11}$$

For the case where $r' = r + \delta r$ with $\delta r << r$ we easily obtain

$$z = \frac{\delta\nu}{\nu} = -\frac{G\,M\,\delta r}{c^2\,r^2}\ . \tag{8.12}$$

In the usual special relativistic approximation[12] the "mass" of the photon is taken to be $h\nu/c^2$ and the change in energy as it moves through a vertical distance δr is

$$\Delta E = \frac{h\nu}{c^2}\,g\,\delta r \tag{8.13}$$

where g is the local gravitational acceleration GM/r^2.

8.3 Comments

Both the preceeding subsections discuss "elementary" situations. In the cosmological case the emitter is at rest in the coordinate system given by the metric (8.6). In physical terms terms it is at rest with respect to the background microwave radiation or equivalently, has no "peculiar" motion. If the emitter does have a peculiar velocity the situation is substantially more complicated. Moreover the usual observers are either earthbound astronomers or the Hubble Telescope which provides a peculiar motion at the receiving end. But, this is quite small relative to other sources of error.

In the case of redshift due to differences in gravitational potential of emitter and observer in a stationary spacetime the situation is more clearly defined. An example is the Pound-Rebka experiment[13] where the difference is precisely known and both emitter and observer are at rest. A different and substantially more complex situation is is presented by the Global Positioning System[14]. Here, the observers are at rest and the emitters are not only at a different gravitational potential, they are moving with high velocity If a metric has Killing vectors in addition to the ones discussed for the radial Doppler effects they can be used for describing Doppler effect due to relative velocities of sources and observers with respect to a distinguished time-like direction given by a (conformal) Killing vector defining a local state of rest.

9 Conservation Theorems

One of the most important properties of Killing vectors is their utility in the derivation of conservation theorems. These are obtained in conjuction with the tangent vectors of geodesics. These are the null vectors for photons and the tangent vectors for force-free point masses. The geodesics are provided by the solutions of the geodesic equation

$$\frac{dk^\alpha}{d\lambda} + \left\{ \begin{array}{c} \alpha \\ \beta \;\; \gamma \end{array} \right\} k^\beta k^\gamma = 0 \tag{9.1}$$

where λ is an *affine parameter* [15] along the trajectory and $k^\alpha = dx^\alpha/d\lambda$. Note that Eq. (9.1) may be written as

$$k^\alpha{}_{;\beta} k^\beta = 0\,. \tag{9.2}$$

Also, for photons and unit masses we have, respectively

$$k^\alpha k_\alpha = 0 \;\text{ and }\; 1\,. \tag{9.3}$$

For a Killing vector, $\boldsymbol{\xi}$ and the tangent vector, $\boldsymbol{k}$, to a geodesic, the product $\mathcal{E} = \xi_\alpha k^\alpha$ is constant along the geodesic. This product is constant because the directional derivative of $\mathcal{E}$ along the geodesic vanishes.

$$\begin{aligned} \dot{\mathcal{E}} &\equiv (k^\alpha \xi_\alpha)_{;\beta} k^\beta \\ &= k^\alpha{}_{;\beta} k^\beta \xi_\alpha + \xi_{\alpha;\beta} k^\alpha k^\beta = 0\,. \end{aligned} \tag{9.4}$$

On the right hand side the first and second terms vanish by virtue, respectively, of the geodesic equation, Eq. (9.2) and the skew-symmetry of $\xi_{\alpha;\beta}$. (See Eq. (3.11).) Consequently, $\dot{\mathcal{E}} = 0$ and $\mathcal{E}$ is constant along the geodesic.

Equation (9.4) is valid for either photons or point masses. For photons there is an additional possibility. If the metric admits either a *homothetic* or *conformal* Killing vector a similar integral exists. The second term in Eq. (9.4) will vanish because $\xi_{\alpha;\beta}$ is symmetric and proportional to $g_{\alpha\beta}$ and k^α is a null vector. In either event the integral is identical in form to that for Killing vector fields, that is, $\mathcal{E} = \xi_\alpha\, k^\alpha = constant$.

Killing vectors are indispensable for the invariant formulation of conservation theorems for fields and extended bodies. Local conservation laws are expressed as the covariant divergence of a symmetric tensor

$$T^{\alpha\beta}{}_{;\beta} = 0\,, \qquad T^{\alpha\beta} = T^{\beta\alpha}\,. \tag{9.5}$$

Given a Killing vector ξ_α define the quantity $S^\beta \equiv \xi_a T^{\alpha\beta}$. We then have

$$S^\beta{}_{;\beta} = \xi_{\alpha;\beta} T^{\alpha\beta} + \xi_\alpha T^{\alpha\beta}{}_{;\beta} = 0\,. \tag{9.6}$$

The first term vanishes because skew-symmetric and symmetric tensors are contracted; the second term vanishes by virtue of the definition of one of its factors. Now, the covariant divergence of a vector may be written as[16]

$$S^\beta{}_{;\beta} = \frac{1}{\sqrt{-g}} (\sqrt{-g} S^\beta)_{,\beta} = 0\,. \tag{9.7}$$

which is a true conservation law.

If, in addition to being symmetric, $T^{\alpha\beta}$ has a vanishing trace, that is $T^{\alpha\beta}g_{\alpha\beta} = 0$, as is the important case of electromagnetic energy-momentum tensor, then conservation laws involving conformal Killing vectors may be obtained. In the first term of Eq. (9.6) substitute in accordance Eq. (5.1). This results immediately in

$$S^{\beta}{}_{;\beta} = \sigma g_{\alpha\beta}T^{\alpha\beta} + \xi_{\alpha}T^{\alpha\beta}{}_{;\beta} = 0\,. \tag{9.8}$$

On the right hand side both terms obviously vanish.

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[e]The following conventions are used: $c = 1$ with metric signature $[1, -1, -1, -1]$; coordinate indices range from $[0-3]$ and in arbitrary coordinate systems are designated by lower case Greek letters, *i.e.,* $[\alpha, \beta, \gamma, \ldots]$*; spatial indices range from* $[1-3]$ *and are designated by lower case Roman case letters* $[i,\ k,\ j,\ \ldots]$*; in Cartesian coordinate systems we use* $[t,\ x,\ y,\ z]$*; partial and covariant differentiation are indicated, respectively, by commas or semicolons, viz.,* $f^{\alpha}{}_{,\beta} = \frac{\partial f^{\alpha}}{\partial x^{\beta}}$ *and* $f^{\alpha}{}_{;\beta} = \frac{\partial f^{\alpha}}{\partial x^{\beta}} + \left\{ \begin{array}{c} \alpha \\ \beta\ \ \gamma \end{array} \right\} f^{\gamma}$.

[1]*A. Einstein, "Über das Relativitäts Prinzip und die aus demselben gezogenen Folgerungen" [On the Relativity Principle and the Conclusions drawn from it], Jahrbuch der Radioaktivität und Elektronik*, **4**, *pp. 411–462 (1907). An English translation is available in vol.2, "Collected Papers of Albert Einstein", translator A. Beck, consultant P. Havas, Princeton University Press, Princeton, NJ, (1989) pp. 252–311.*

[2]*A. Einstein, " Die Grundlage der allgemeinen Relativitätstheorie, Annalen der Physik*, **49**, *pp. 769–822 (1916). An English translation is contained in vol. 6 "Collected Papers of Albert Einstein", translator A. Engel, consultant E. L. Schucking, Princeton University Press, Princeton, NJ, (1997) pp. 147–200.*

[3]*H. Weyl, Raum-Zeit-Materie, 5th edition, Springer-Verlag, Berlin (1923). This was edited and annotated by Juergen Ehlers and reprinted as the 7th edition, Springer-Verlag, Berlin (1988). The English translation is by E. L. Schucking.*

[4]*See, e.g., R. d'Inverno, Introducing Einstein's Relativity, Clarendon Press, Oxford (1992), Section 7.7.*

[5]*S. W. Carroll, Spacetime and Geometry, Addison Wesley, New York (2004). See section 3.8.*

[6] *In 3-dimensional elasticity theory this tensor is known as the strain tensor due to an infinitesimal displacement* $\epsilon f^i(x^j)$. *See, e.g., F. Ziegler, Mechanics of Solids and Fluids, Springer-Verlag, New York (1991). See section 1.3, "Kinematics of Deformable Bodies".*

[7] *W. Rindler, Relativity - Special, General, Cosmological, Oxford University Press, New York (2001). See sections 16.4 and 16.5.*

[8] *S. W. Carroll, loc cit, section 8.2.*

[9] *A note by T. Rothman discussing its genesis and content has appeared as a "Golden Oldie" in General Relativity and Gravitation,* **34**, *pp.1541–1543 (2002).*

[10] *R. d'Inverno, loc cit, section 14.6.*

[11] *S. W. Carroll, loc cit, section 5.2.*

[12] *W. Rindler, loc cit, pp. 4–26.*

[13] *R. V. Pound and G. A. Rebka, "Apparent Weight of Photons", Physical Review Letters,* **4**, *pp. 337–341 (1960).*

[14] *N. Ashby, "Relativity and the Global Positioning System", Physics Today, May 2002 pp. 41-47; "Relativity in the Global Positioning System", Living Reviews in Relativity, http://www.livingreviews.org/lrr–2003–1, Max-Planck-Gesellschaft, Potsdam, Germany (2003).*

[15] *S. W. Carroll, loc cit, section 3.4*

[16] *S. W. Carroll, loc cit, see p. 101.*

Figure 1: Weyl's Geometry of the Doppler Effect
The points P and P' on the worldline of a point-like light source are the origin of surfaces of constant phase that form future null cones. The worldline of the observer intersects these two light cones in points Q and Q'. The light signals PQ and $P'Q'$ are given by null geodesics in the ray approximation. We draw the light cones as having "straight" sides, but that is strictly a convention; the intervening spacetime could be curved.

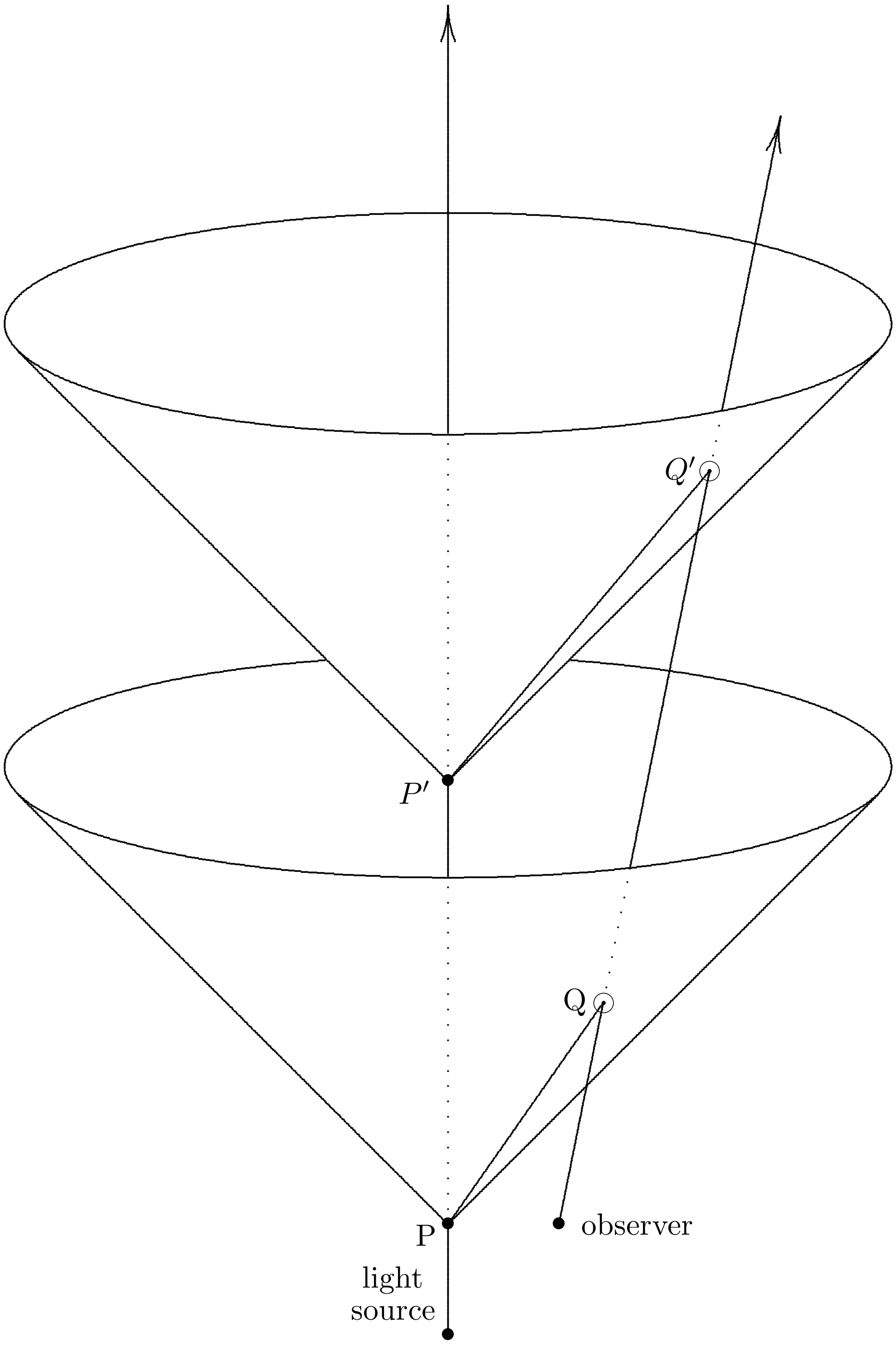
Figure 2: Doppler Geometry
The notation is the same as in Figure 1. If the light-emitting process in the source has the infinitesimal period lasting from events P to P' it will be perceived by the observer of having a period lasting from Q to Q'. The ratio of their proper times ds'/ds is the redshift $1+z$.

Figure 3: Killing Motion
The vector field ξ^α moves two neighboring points P and Q by $\epsilon\xi^\alpha$ into the points P' and Q'. This will in general change the distance ds of the infinitesimal interval $\overline{PQ}$ into the distance $\overline{ds}$ of $\overline{P'Q'}$. If $\overline{ds} = ds$ for all neighboring points Q, then ξ^α is a Killing vector field.

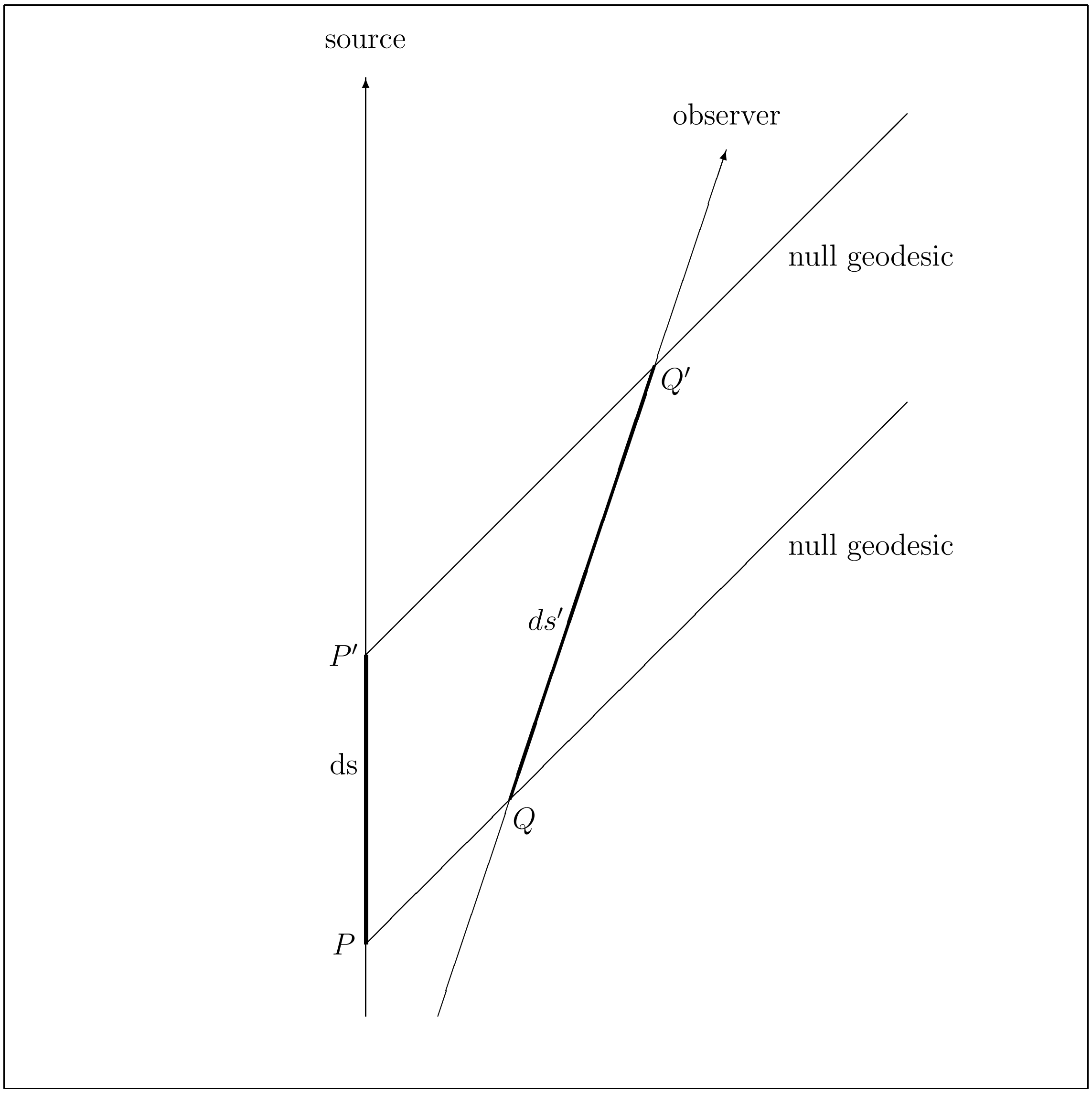
Figure 4: A Conformal Vector Field for Source and Observer
The conformal Killing vector field moves the null geodesic connecting $P(x)$ with $Q(y)$ into another null geodesic connecting P' with Q'. If source and observer move on conformal Killing lines, the redshift $1+z$ is given by the ratio of the length of the Killing vectors $|\xi(y)|/|\xi(x)|$.

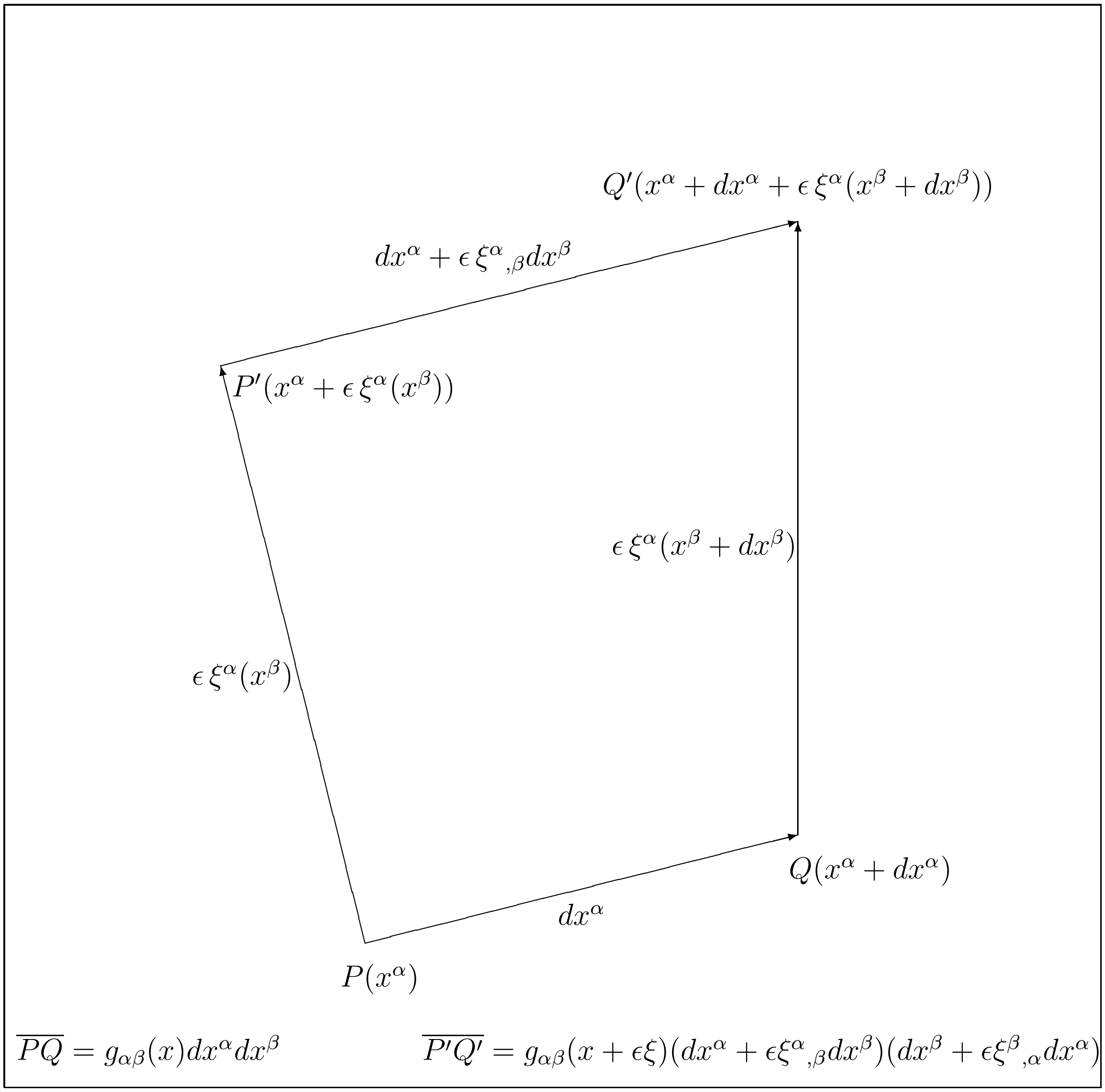
Figure 5: Radial Doppler Shift in Minkowski Spacetime
Coordinates are chosen to assume source and observer are in a 4-dimensional plane with the source as time axis. A light signal emitted by the source at time t is received by the observer at distance x and time t'. With speed of light $= 1$, we have $x = t' - t$.

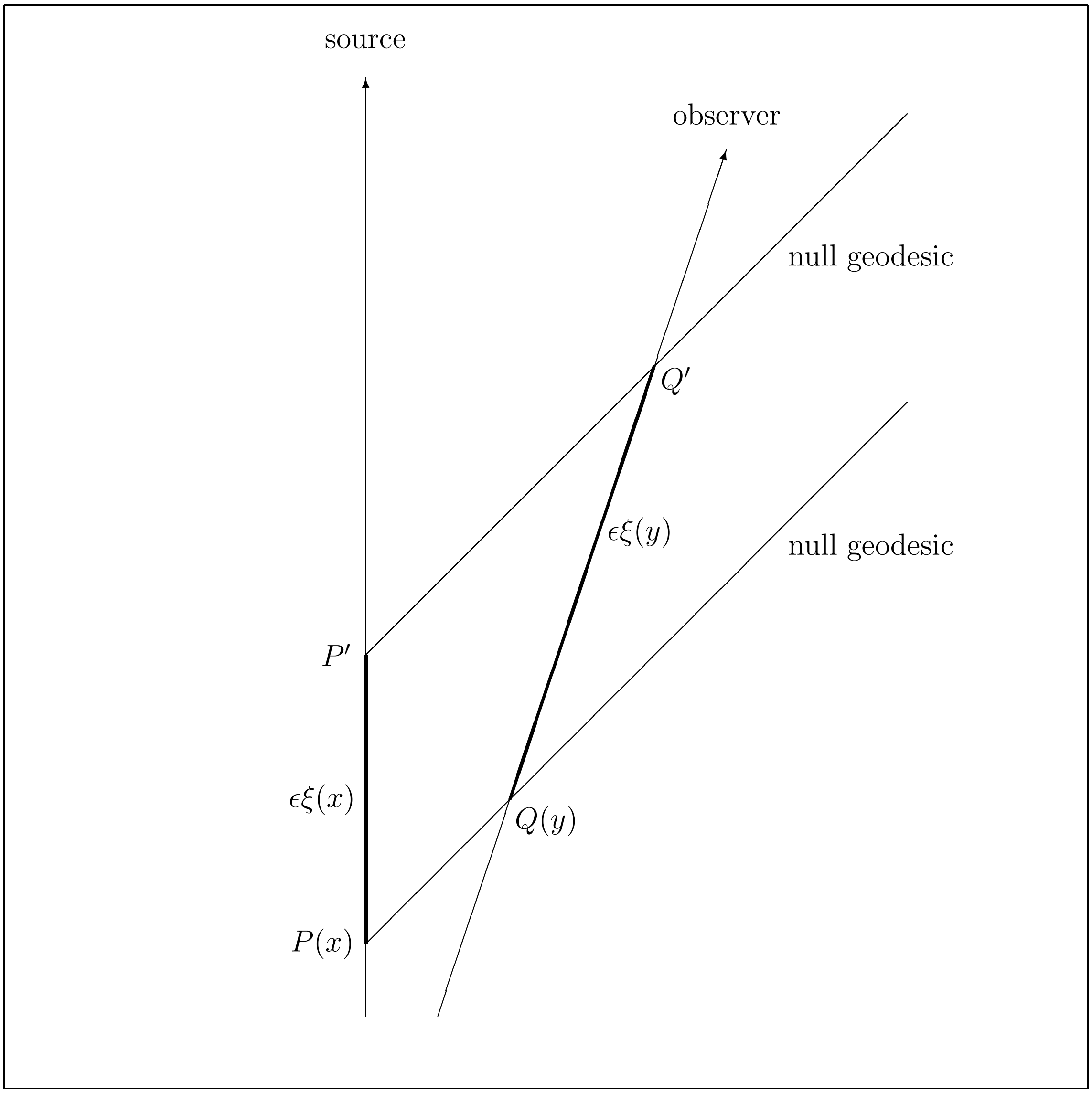
Figure 6: 4-dimensional Polar Coordinates for Spacetime
Straight time-like world lines through the origin lie at constant rapidity χ. The transformation $t = T\cosh\chi$, $r = \sinh\chi$ gives $T^2 = t^2 - r^2$. T measures proper time along the rays originating from the origin. T is also the length of the homothetic Killing vector.

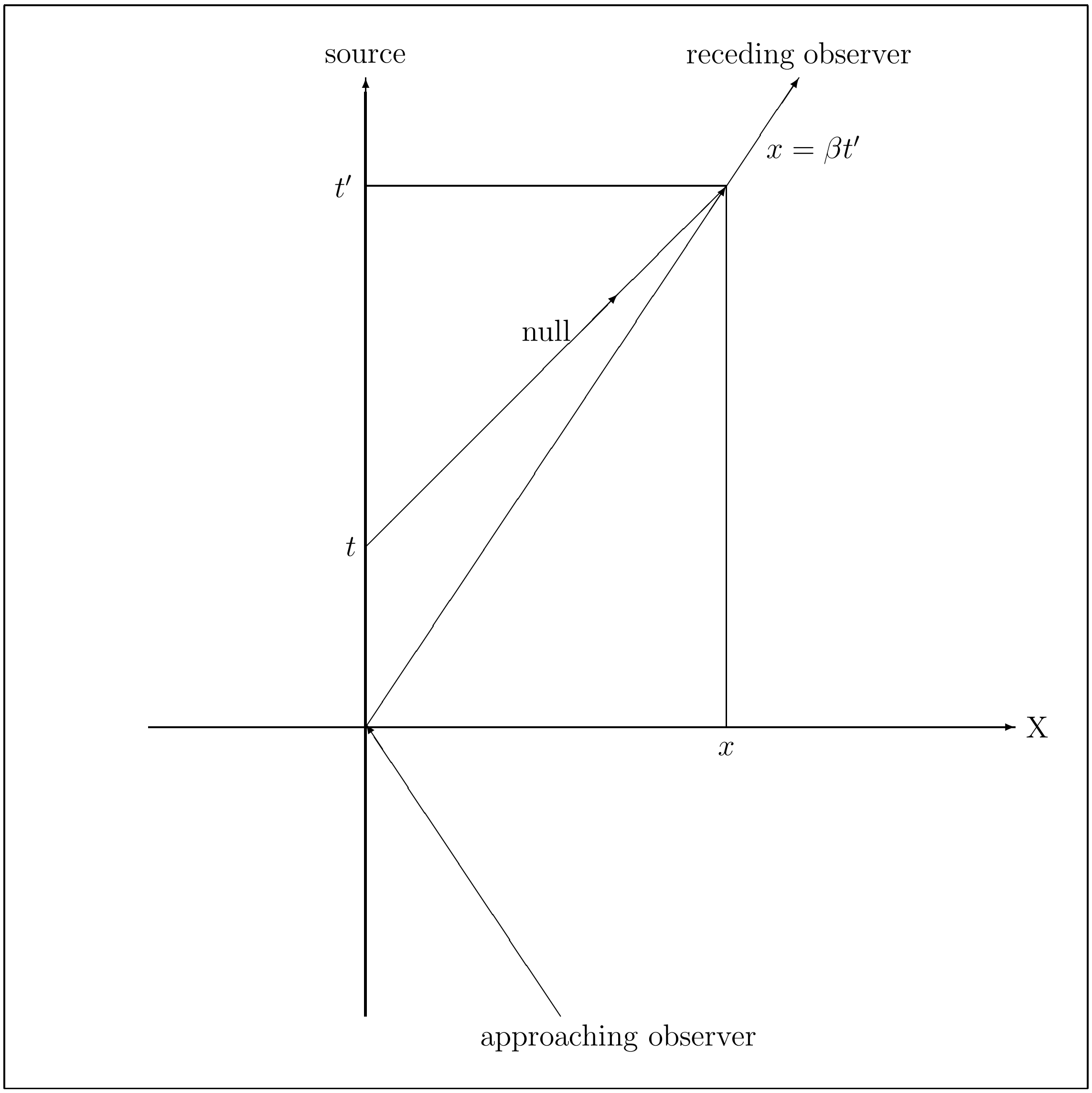


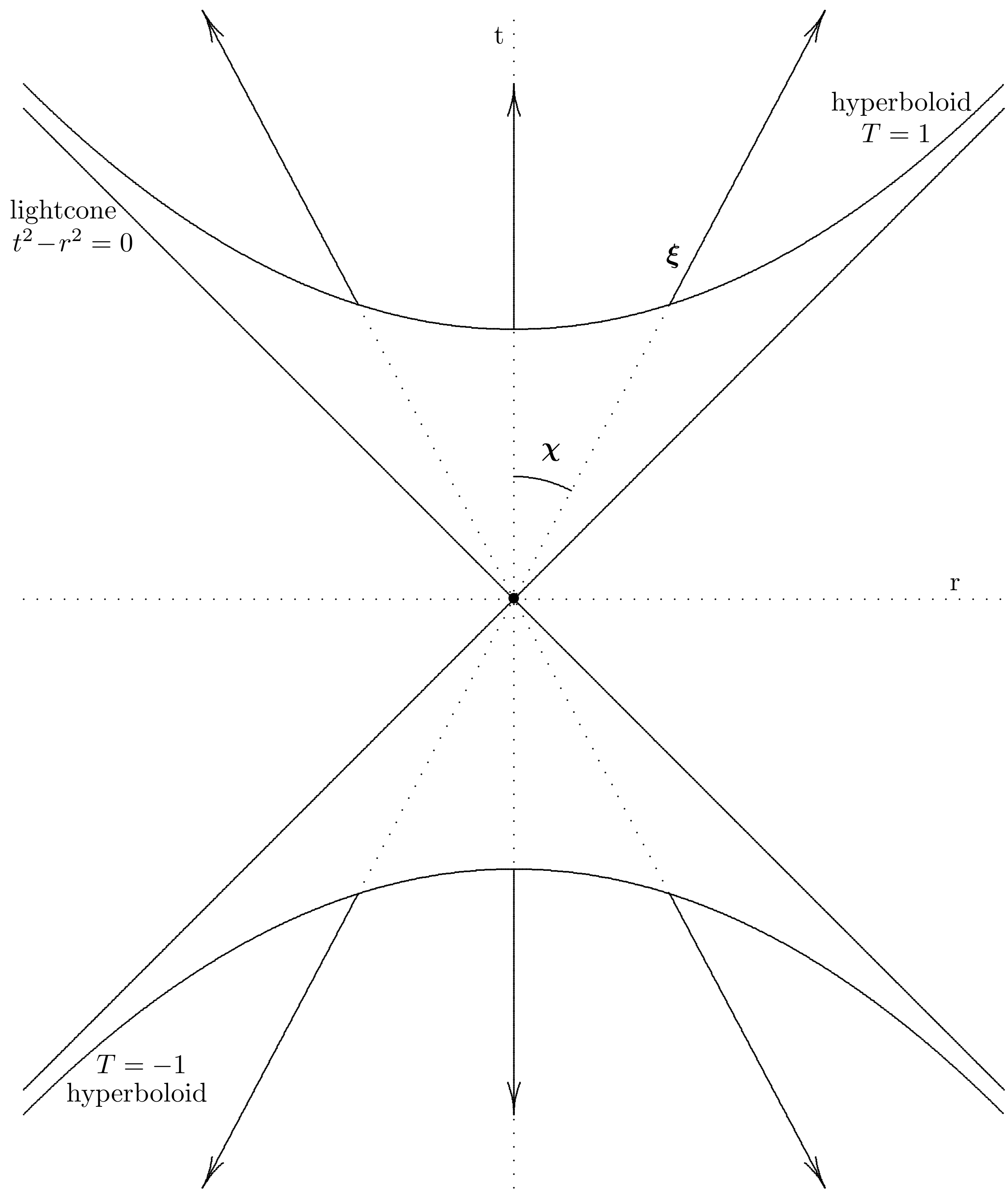
Q'
P'
Q
P
observer
light
source


source
observer
null geodesic
Q'
null geodesic
ds'
P'
ds
Q
P


$Q'(x^\alpha + dx^\alpha + \epsilon\,\xi^\alpha(x^\beta + dx^\beta))$
$dx^\alpha + \epsilon\,\xi^\alpha{}_{,\beta} dx^\beta$
$P'(x^\alpha + \epsilon\,\xi^\alpha(x^\beta))$
$\epsilon\,\xi^\alpha(x^\beta + dx^\beta)$
$\epsilon\,\xi^\alpha(x^\beta)$
$Q(x^\alpha + dx^\alpha)$
dx^α
$P(x^\alpha)$
$\overline{PQ} = g_{\alpha\beta}(x) dx^\alpha dx^\beta$
$\overline{P'Q'} = g_{\alpha\beta}(x + \epsilon\xi)(dx^\alpha + \epsilon\xi^\alpha{}_{,\beta} dx^\beta)(dx^\beta + \epsilon\xi^\beta{}_{,\alpha} dx^\alpha)$


source
observer
null geodesic
Q'
$\epsilon\xi(y)$
null geodesic
P'
$\epsilon\xi(x)$
$Q(y)$
$P(x)$


source
receding observer
$x = \beta t'$
t'
null
t
X
x
approaching observer


t
hyperboloid
$T=1$
lightcone
$t^2-r^2=0$
ξ
χ
r
$T=-1$
hyperboloid