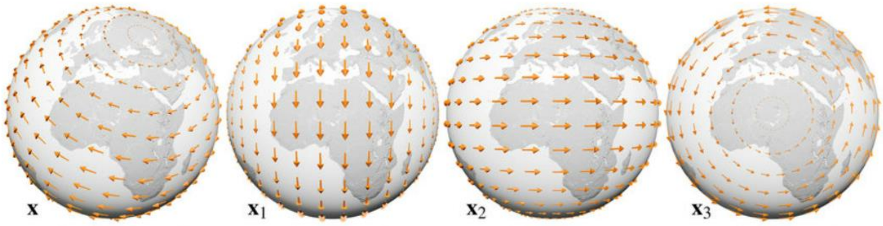
Killing vector fields

§ 04



$$X=\begin{pmatrix}0 & -c & b\\ c & 0 & -a\\ -b & a & 0\end{pmatrix}\quad X_1=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\quad X_2=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\quad X_3=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Killing vector fields give the infinitesimal isometries of a manifold M，here the sphere S^2 。

X_1, X_2, X_3即是 Lie algebra so(3)中的無窮小生成元(infinitesimal generators)，

$[X_i, X_j]=\varepsilon_{ijk}X_k$ ，ε_{ijk} 稱為 Levi-Civita symbols

$$\frac{d}{d\theta}\Big|_{\theta=0}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=-y\partial_x+x\partial_y$$

The number of isometries is the number of linearly independent Killing vector fields。

We refer to an n-dimensional manifold with $\dfrac{n(n+1)}{2}$ 。

Killing vectors as a maximally symmetric space。

Symmetry and Lie algebra

We expect S^2 to have symmetry under the action of SO(3)。

$Z=-y\partial_x+x\partial_y=\partial\varphi$ ，$X=-z\partial_y+y\partial_z$ ，let $[Z,X]=-z\partial_x+x\partial_z=Y$

Then $[X,Y]=Z,[Y,Z]=X$

Span{X, Y, Z} = Lie algebra so(3)

X、Y、Z are the Killing fields that generate so(3)。

Since $L_{[X,Y]}=L_XL_Y-L_YL_X=0$ ，we can find a third Killing vector by taking the commutator of the first two，given that X and Y are independent Killing vector fields，i.e. $[X,Y]\neq 0$

The Lie algebra elements X，$X_i\in$ so(3) generate the Killing fields X, x_i on M through their Lie algebra action。

Like the matrices X_i，the Killing fields x_i are linearly independent，forming a basis of the Lie algebra of Killing fields on S^2。

That is，as we can expand $X = aX_1 + bX_2 + cX_3$，we likewise get

$x = ax_1 + bx_2 + cx_3$，where the latter means point-wise addition of vectors in each $T \times M$ at each x ∈ M。

Hidden Symmetries of Dynamics in Classical and Quantum Physics by Marco Cariglia

Example

Consider the usual Euclidean space $(R^3, dx^2 + dy^2 + dz^2)$，there are at most 6 linearly independent Killing fields。

- $T_s(x, y, z) = (x + s, y, z)$ is a 1-parameter family of isometries, since we have that $DT_s(x, y, z) = \mathrm{Id}_{\mathbb{R}^3}$. Thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} T_s(x, y, z) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (x + s, y, z) = \partial_x$$

  is a Killing field (we already knew that). Similarly we recover that ∂_y and ∂_z are Killing fields.