(M,g) is a Riemann manifold。ξ is a Killing vector field if

$(L_\xi g)(X,Y)=g(\nabla_X\xi,Y)+g(\nabla_Y\xi,X)=0$ for all $X,Y\in\chi(M)$

Simply $L_X g=0$ or $\nabla_\mu K^\nu=-\nabla_\nu K^\mu$

§ Isometry

(M,g) is a Rieannian manifold with Levi-Civita connection ∇。

ϕ:M $\to$ M is an isometry $\Leftrightarrow \phi^* g_{\phi(p)}=g_p$

$g_{\phi(p)}(\phi_* X,\phi_* Y)=g_p(X,Y)$ for all $X,Y\in T_pM$

Or in coordinates $\dfrac{\partial y^\alpha}{\partial x^\mu}\dfrac{\partial y^\beta}{\partial x^\nu}g_{\alpha\beta}(\phi(p))=g_{\mu\nu}(p)$

Where the diffeomorphism ϕ is an isometry (討論 GR 中的等距同構)，preserve the metric。

The flows generated by Killing fields are continuous isometries on the manifold。

Isometries naturally form a group。

A Killing vector satisfies $\nabla_{(\mu K\nu)}=0$ ，and that implies that $K_\nu p^\nu$ is conserved along a geodesic。 p^ν is a 4-momentum。

§ Definition of Killing vector field

$\varphi_t : M\to M$ is a one-parameter group of isometries， $X\in\chi(M)$

$X_p := \dfrac{d}{dt}\Big|_{t=0}\varphi_t(p)$ is called the Killing vector field associated to φ_t

$X,Y\in\chi(M)$ ， $L_X Y := \dfrac{d}{dt}((\varphi_{-t})_* \mathrm{Y})_{t=0}$ ，where $\{\varphi_t\}_{t\in I}$ is the flow of X。

And $L_X Y=[X,Y]$ ， $L_X\omega := \dfrac{d}{dt}(\varphi_t^{\,*}\omega)\Big|_{t=0}$

$\nabla_\mu \mathrm{V}^\nu=\partial_\mu \mathrm{V}^\nu+\Gamma^\nu_{\mu\lambda}\mathrm{V}^\lambda$ and $\nabla_\mu\omega_\nu=\partial_\mu\omega_\nu-\Gamma^\lambda_{\mu\nu}\omega_\lambda$

1. If a Killing field X generates isometries then $L_X g = 0$。

   And since $L_{[X,Y]} = L_X L_Y - L_Y L_X = 0$，we can find a third Killing vector by taking the commutator of the first two，given that X and Y are independent Killing vector fields，i.e. $[X,Y] \neq 0$

   Manifold 的 metric 在這組向量的方向上保持不變。(preserves the metric。)

   Flows generated by Killing fields are continuous isometries on the manifold。

2. A vector field $K = K^{\mu}\partial_{\mu}$ on M is said to be a Killing vector field if the infinitesimal displacement $\varphi : x^{\mu} \to x^{\mu} + \varepsilon K^{\mu}$ generates an isometry。

   Show that this is the case，if $X^{\kappa}\partial_{\kappa} g_{\mu\nu} + \partial_{\mu} X^{\kappa} g_{\kappa\nu} + \partial_{\nu} X^{\kappa} g_{\mu\kappa} = 0$

   These are the so-called Killing equation。

3. Show that the Killing equations can be written as

   $L_X g_{\mu\nu} = \nabla_{\mu} X_{\nu} + \nabla_{\nu} X_{\mu} = 0$

   $(L_X g)_{\mu\nu} = X^{\rho}\partial_{\rho} g_{\mu\nu} + (\partial_{\mu} X^{\rho}) g_{\rho\nu} + (\partial_{\nu} X^{\rho}) g_{\mu\rho} = 0$

4. $\nabla_{\mu} X^{\nu} + \nabla_{\nu} X^{\mu} = 0$ is the Killing equation for X^{μ}

5. (c) Work out the Killing vector fields of the Minkowski spacetime $(\mathbb{R}^{3,1}, \eta)$ by solving the Killing equations. *(3 points)*

6. (d) Show that an n-dimensional Minkowski space ($n \geq 2$) is equipped with $n(n+1)/2$ Killing vector fields. Spaces which admit this number of Killing vector fields are called *maximally symmetric spaces.* *(1 point)*

7. (e) Verify that if the metric is independent of some coordiante x^{σ} the corresponding vector ∂_{σ} is a Killing vector. *(2 points)*

(f) Let X^μ and Y^ν be two Killing vector fields. Show that any linear combination of those two vectors is a Killing vector field and that the Lie-bracket $[X, Y]$ is a Killing
8. vector field as well. Conclude that the set of Killing vector fields forms a Lie algebra.

Now, as an example we consider the two-sphere S^2 with its usual induced metric.

9. (g) Write down the three Killing equations for the vector field $X = X^\theta \partial_\theta + X^\varphi \partial_\varphi$. (*1 point*)

(h) Show that X_θ is independent of θ. We may write $X_\theta(\theta, \varphi) = f(\varphi)$. By substituting this into one of the Killing equations work out that X_φ satisfies

$$X_\varphi = -F(\varphi) \sin\theta \cos\theta + g(\theta)\,,$$

10. where $F(\varphi)$ is the primitive of $f(\varphi)$ and $g(\theta)$ is some integration constant. (*1 point*)

(i) By plugging the result of the previous task into the last remaining Killing equation show that by separation of variables one obtains

$$\frac{\mathrm{d}g}{\mathrm{d}\theta} - 2g(\theta)\cot\theta = C\,, \tag{1}$$

$$\frac{\mathrm{d}f}{\mathrm{d}\varphi} + F(\varphi) = -C\,, \tag{2}$$

for some constant C. (*2 points*)

11.

(j) By integrating (1) find $g(\theta)$. By differentiating (2) show that f is harmonic and write down the general solution. You should end up with

$$g(\theta) = (C_1 - C\cot\theta)\sin^2\theta\,, \tag{3}$$

12.

$$f(\varphi) = A\sin\varphi + B\cos\varphi\,. \tag{4}$$

(k) Putting all results together show that a general Killing vector on S^2 is given by

13.

$$X = A\,(\sin\varphi\,\partial_\theta + \cos\varphi\cot\theta\,\partial_\varphi) + B\,(\cos\varphi\,\partial_\theta - \sin\varphi\cot\theta\,\partial_\varphi) + C_1\,\partial_\varphi\,. \tag{5}$$

(l) Identify the three basis vectors of the Killing vector (5) with the angular momentum $L_i = \sum_{j,k} \epsilon_{ijk} x_j \partial_k$. Argue that the Killing vectors on S^2 generate the Lie algebra $\mathfrak{so}(3)$. Is S^2 a maximally symmetric space? (*2 points*)

14.

Riemann tensor $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Torsion $\mathrm{T(X,Y)} = \nabla_{\mathrm{X}} \mathrm{Y} - \nabla_Y X - [X,Y]$

$R_{\sigma\mu\nu}^{\ \ \ \ \rho} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}$

$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda_{\sigma\mu\nu}$

$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$ ， $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$ ， $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$ ， $R_{\rho[\sigma\mu\nu]} = 0$

Bianchi Identity :

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_\beta \partial_\mu g_{\nu\alpha} + \partial_\alpha \partial_\nu g_{\beta\mu} - \partial_\beta \partial_\nu g_{\alpha\mu} - \partial_\alpha \partial_\mu g_{\beta\nu})$$

$R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0$ L.Bianchi 1902

Where $R^h_{ijk,l}$:=covariant derivative of R^h_{ijk} w.r.t. the $l-th$ coordinate。

M is a manifold with symmetric connection $\nabla\,(\nabla_X Y - \nabla_Y X = [X,Y])$，then

$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

Show that the sum of cyclic permutations of the last three indices of the curvature tensor vanishes, i.e.

$$R_{\kappa\lambda\mu\nu} + R_{\kappa\mu\nu\lambda} + R_{\kappa\nu\lambda\mu} = 0\,, \qquad 1^{\text{st}}\ \text{Bianchi identity}\,. \tag{6}$$

Make use of locally inertial coordinates once more to prove

$$\nabla_{[\mu} R_{\kappa\lambda]\rho\sigma} = 0\,, \qquad 2^{\text{nd}}\ \text{Bianchi identity}\,. \tag{7}$$

By contracting indices of the second Bianchi identity (7) twice, show that

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R\,.$$

Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$

Scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$

Torsion tensor $T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]}$ then $[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} \nabla_\lambda V^\rho$

$\nabla_\mu \nabla_\nu V^\rho = \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu} \nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma} \nabla_\nu V^\sigma = \ldots.$

§ Examples

1. On R^3，$g = dx^2 + dy^2 + dz^2$

   There are $\frac{1}{2} \times 3 \times 4 = 6$ linearly independent Killing vector fields

   一. Translations

   $T_s(x, y, z) = (x + s, y, z)$ is a 1-parameter family of isometries

   $DT_s(x, y, z) = Id_{R^3}$

   $\frac{d}{ds}\big|_{s=0} T_s(x, y, z) = \frac{d}{ds}\big|_{s=0} (x + s, y, z) = \partial_x$，so as ∂_y, ∂_z

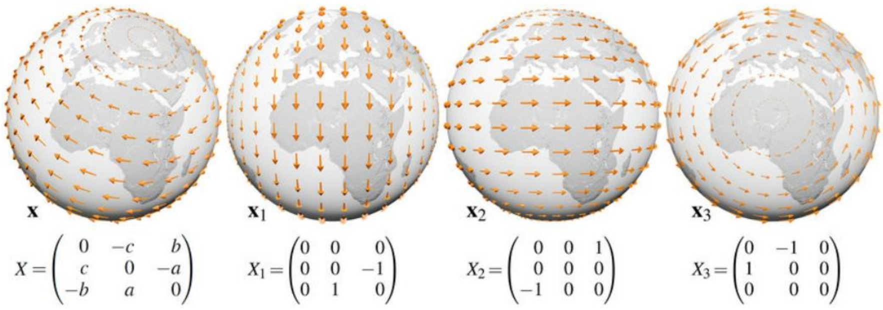
   二. Rotations

   One-parameter family of rotation around the x-axis

$$\frac{d}{d\theta}\Big|_{\theta=0} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -y\partial_x + x\partial_y$$

   And similarly $-z\partial_x + x\partial z$，$-z\partial_y + y\partial_z$

2. Killing field on S^2



Killing vector fields give the infinitesimal isometries of a manifold M，here the sphere S^2。

We expect S^2 to have symmetry under the action of SO(3)。

$$S^2 : \begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases}$$

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$$

$\frac{\partial}{\partial\varphi}$ ：vector field which generates rotation about the z-axis。

A rotation about any axis is a isometry，so ∂_φ is a Killing vector。

∂_θ is not a Killing vector。

$Z = x\partial_y - y\partial_x = \partial\varphi$ ， $X = z\partial_y - y\partial_z$ ， $Y = z\partial_x - x\partial_z = [X, Z]$

And $[X,Y] = Z, [Y,Z] = X$

Span{X, Y, Z} = Lie algebra so(3)

In terms of spherical coordinates gives

$X = \sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi$ ， $Y = \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi$

X、Y、Z are the Killing fields that generate so(3)。

The Lie algebra elements X， X_i ∈ so(3) (bottom) generate the Killing fields x, x_i on M (top) through their Lie algebra action。

Like the matrices X_i ，the Killing fields x_i are linearly independent，forming a basis of the Lie algebra of Killing fields on S^2 。

That is，as we can expand $X = aX_1 + bX_2 + cX_3$ ，we likewise get

$x = ax_1 + bx_2 + cx_3$ ，where the latter means point-wise addition of vectors in each $T \times M$ at each x ∈ M。

3. For the metric $ds^2 = A(z)^2(-dt^2 + dx^2 + dy^2) + dz^2$

   $T := \frac{\partial}{\partial t}, X := \frac{\partial}{\partial x}, Y := \frac{\partial}{\partial y}$ are Killing vectors。

   Hint check that $(L_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu X^\rho + g_{\rho\mu}\partial_\nu X^\rho = 0$

§ Exercises

1. If $c: I \to M$ is a geodesic，then $<\dot{c}(t), X_{c(t)}>$ is constant。

2. Let ξ be a Killing vector field of constant length，prove that the integral curves of ξ are geodesics of (M ,g)。

   The hypothesis says that $X\langle \xi, \xi \rangle = 0$ for all $X \in \chi(M)$

   Then $2 < \nabla_X \xi, \xi >= 0$ ，by Killing equation

   $(L_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0$

   $(L_\xi g)(\xi, X) = <\nabla_\xi \xi, X> + <\xi, \nabla_X \xi> = 0$

   $< \nabla_\xi \xi, X >= 0$ for all $X \in \chi(M)$ ，$\nabla_\xi \xi = 0$ ，it means the integral curves of ξ are geodesics。

3. Consider the Killing vector fields on $M = S^2$ with metric $g = d\theta^2 + \sin^2\theta d\phi^2$

   $K_1 = \partial_\phi$ ，$K_2 = -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi$ ，$K_3 = \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi$

   Verify that they satisfy the Killing equations。

4. For any Killing field ξ ，$div\xi = 0$

   Proof: The right side of coordinate expression $\xi^j{}_{;j} = g^{ij}\xi_{i;j}$ for the divergence of ξ is both symmetric and skew-symmetric in i and j, hence vanishes. □

   Remark. Another proof is noting that div ξ is the trace of the skew-symmetric endomorphism $\nabla\xi$.

5. If ξ is Killing then，$\nabla_X(\nabla\xi) = R(X,\xi)$ for all X

6. Show that any Killing vector K^μ satisfies (1) $\nabla_\mu\nabla_\sigma K^\rho = R^\rho_{\sigma\mu\nu}K^\nu$ (2) $K^\lambda\nabla_\lambda R = 0$

   Hint (1) derivatives of Killing vectors (2) along with the Bianchi identity and Killing equation and (1)

(1) Killing equation $\nabla_\sigma K^\rho = \nabla_\rho K^\sigma$

$$[\nabla_\mu, \nabla_\sigma]K^\rho = (\nabla_\mu \nabla_\sigma - \nabla_\sigma \nabla_\mu)K^\rho$$

$$[\nabla_\rho, \nabla_\sigma]K^\mu = (\nabla_\rho \nabla_\sigma - \nabla_\sigma \nabla_\rho)K^\mu = \nabla_\rho \nabla_\sigma K^\mu + \nabla_\sigma \nabla_\mu K^\rho$$

$$[\nabla_\rho, \nabla_\mu]K^\sigma = (\nabla_\rho \nabla_\mu - \nabla_\mu \nabla_\rho)K^\sigma = -\nabla_\rho \nabla_\sigma K^\mu + \nabla_\mu \nabla_\sigma K^\rho$$

Then $\nabla_\mu \nabla_\sigma K^\rho = \frac{1}{2}([\nabla_\mu, \nabla_\sigma]K^\rho + [\nabla_\rho, \nabla_\sigma]K^\mu + [\nabla_\rho, \nabla_\mu]K^\sigma)$

$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\sigma\mu\nu}V^\sigma - T^\lambda_{\mu\nu}\nabla_\lambda V^\rho$, where $T^\lambda_{\mu\nu}$ is the torsion tensor 。

$$\nabla_\mu \nabla_\sigma K^\rho = \frac{1}{2}\{R^\rho_{\nu\sigma\mu}K^\nu + R^\mu_{\nu\sigma\rho}K^\nu + \mathrm{R}^\sigma_{\nu\mu\rho}\mathrm{K}^\nu\}$$

(2)

d) Consider a geodesic $\mathcal{C}$ parameterized by λ, where λ is affinely related to the arc length. Deduce from the defintion (4) that for a Killing vector field K

$$g\left(K, \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}\right) = \text{constant along the geodesic } \mathcal{C}\ . \tag{7}$$

7. These are first order differential equations for the geodesic. *(2 points)*

e) On the exercise sheet 4, we have introduced the Lie bracket (commutator) of vector fields, denoted as $[X, Y]$. Show that

$$[K_i, K_j] = -\epsilon_{ijk}K_k\ , \quad i, j, k = 1, 2, 3\ , \tag{8}$$

8. with the totally antisymmetric ϵ-tensor that is normalized by $\epsilon_{123} = 1$. *(3 points)*

f) Note that the vector fields $-iK_j$ for $j = 1, 2, 3$ and $i^2 = -1$ fulfill the angular momentum algebra. Can you think of a reason for this? *(2 points)*

g) For the Killing vector fields given in eq. (5), write eqs. (7) in terms of the local coordinates. Label the constant appearing on the right hand side of (7) for $K = K_i$ as L_i *(3 points)*

h) Combine the equations found in the previous item to arrive at an equation, in which θ, ϕ and the L_i but no derivatives of the geodesic appear. This equation can (at least locally) be solved to yield the geodesics in the form $\theta(\phi)$ or $\phi(\theta)$. *(2 points)*

9. i) Find the geodesics for the three cases in which only one L_i is non-zero. *(2 points)*

Proposition. *Let ξ be a Killing field and assume that $\xi = \operatorname{grad} f$ for some smooth function $f\colon M \to \mathbb{R}$. Then $\nabla\xi = 0$ and $\triangle f = 0$.*

Proof: In the conditions of the statement, we have that $\langle \nabla_X \xi, Y\rangle = \operatorname{Hess}(f)(X,Y)$ for any vector fields $X, Y \in \mathfrak{X}(M)$. This is skew-symmetric in X and Y since $\nabla\xi$ is skew-adjoint. On the other hand, it is also symmetric, since torsion-free connections produce symmetric Hessian tensors. So $\langle \nabla_X\xi, Y\rangle = 0$ for all Y implies that $\nabla_X \xi = \mathbf{0}$ for all X, and so $\nabla\xi = 0$. On the other hand, since $\operatorname{Hess}(f) = 0$, taking the trace we obtain $\triangle f = 0$ as well. □

10.

Proposition. *Assume that M is compact, Riemannian and oriented. If ξ is a Killing field and there is a smooth function $f\colon M \to \mathbb{R}$ such that $\xi = \operatorname{grad} f$, then f is constant and $\xi = \mathbf{0}$.*

Proof: We have that $\triangle f = \operatorname{div}\operatorname{grad} f = \operatorname{div}\xi = 0$, so that[3]

$$\triangle(f^2) = 2f\triangle f + 2\|\operatorname{grad} f\|^2 = 2\|\xi\|^2,$$

and thus

$$0 \overset{(*)}{=} \int_M \triangle(f^2)\,\mathrm{d}M = \int_M 2\|\xi\|^2\,\mathrm{d}M \implies \|\xi\| = 0 \implies \xi = \mathbf{0},$$

as wanted, where $(*)$ follows from Stokes' Theorem and the general divergence expression[4] in terms of the volume form $\mathrm{d}M$: $\mathrm{d}(\iota_X \mathrm{d}M) = (\operatorname{div} X)\,\mathrm{d}M$ for any vector field $X \in \mathfrak{X}(M)$. □

11.

12. Let M be a compact Riemannian manifold of even dimension whose sectional curvature is positive。Prove that every Killing field X on M has a singularity (there exists a $p \in M$ such that X(p)=0)

Hint: Let $f\colon M \to R$ be the function $f(q) = \langle X, X\rangle(q)$, $q \in M$, and let $p \in M$ be a minimum point of of f (Cf. the previous Exercise). Suppose that $X(p) \neq 0$. Define a linear mapping $A\colon T_pM \to T_pM$ by $A(y) = A_X Y = \nabla_Y X$, where Y is an extension of $y \in T_pM$. Let $E \subset T_pM$ be orthogonal to $X(p)$. Use the previous exercise to show that $A\colon E \to E$ is an anti-symmetric isomorphism. This implies that $\dim E = \dim M - 1$ is even, which is a contradiction; thus $X(p) = 0$.

Killing tensors are not related in a simple way to symmetries of the spacetime，but they will simplify our analysis of rotating black holes and expanding universes。

Killing vector fields on a manifold are in one-to-one correspondence with continuous symmetries of the metric on the manifold. Every Killing vector implies the existence of conserved quantities associated with geodesic motion. Physically this can be understood in the following way: A particle moving along the direction of the Killing vector will not feel forces and the component of momentum in that direction will be conserved.

The number of isometries is given for a number of linearly independent Killing vector fields, i.e. the number of independent solutions of the Killing equations (3). We refer to an n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vectors as a *maximally symmetric space*. This happens to be the case for Euclidean spaces $\mathbb{R}^n$ and n-spheres S^n. Other maximally symmetric spaces of interest in cosmology are the de Sitter $\mathrm{d}S_n$ and Anti-de Sitter $\mathrm{Ad}S_n$ geometries, which are Lorentzian manifolds with positive and negative curvature respectively.

§ maximally symmetric spaces

§ Gravitational redshift and Killing vectors https://arxiv.org/abs/gr-qc/0508125

§ Lie group symmetry

§ Noether theorem

§ 參考資料

[Notes on Killing fields] https://www.asc.ohio-state.edu/terekcouto.1/texts/killing.pdf

[Youtube] https://www.youtube.com/watch?v=ZXrwhhQAEss