§ Conjugate points and the singularities of the exponential map

$\gamma:[0,l]\to M$ 是一測地線，$\gamma(0)=P$ ，$Q=\gamma(t_0)$　$t_0\in[0,l]$

若存在一沿著γ 不全為 0 的 Jacobi 場 J(t)，使得$J(0)=J(t_0)$，則稱 Q 是 P 的共軛點(關於測地線γ)。因此，上例中 球面上 Q 是 P 的共軛點。

Let $\gamma:[0,l]\to M$ be a geodesic and $\gamma(0)=p$ 。

The point $q=\gamma(t_0)$ is conjugate to p along γ if and only if $v_0=t_0\gamma'(0)$ s a critical point of $\exp_p$ 。

In addition，the multiplicity of q as a conjugate point of p is equal to the dimension of the kernel of the linear map $(d\exp_p)_{v_0}$ 。

§ And the maximum number of such linearly independent fields is called the multiplicity of the conjugate point to $\gamma(t_0)$

The order of multiplicity of conjugacy is the dimension of the space of Jacobi fields vanishing at p and q。

§ Theorem

Let γ be a smooth timelike curve connecting two points $p,q\in M$ 。

Then the necessary and sufficent condition that γ locally maximizes proper between p and q over smooth one parameter variations is that γ be a geodesic with no point conjugate to p between p and q。

The theorem is the basis for the singularity theorem of GR。