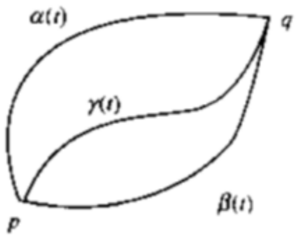
§ Jacobi 場與變分法的關係

Let M is a Riemann manifold with symmetric connection ∇

$c:[a,b]\to M$ a geodesic

$H(t,u):[a,b]\times(-\varepsilon,\varepsilon)\to M$ a variation of c with



H(t,0)=c(t)，H(a,u)=c(a)，H(b,u)=c(b)，and is a geodesic as well

Let $J(t)=\dfrac{\partial H}{\partial u}$ variation vector field (類似切向量)

$\dot{c}(t)=\dfrac{\partial H}{\partial t}$ velocity vector field along c

($J=\dfrac{\partial}{\partial u},V=\dfrac{\partial}{\partial t}$ ，then $L_J V=[J,V]=0$

Torsion free ，$T(J,V)=\nabla_J V-\nabla_V J-[J,V]=0$ then $\nabla_J V=\nabla_V J$

$$R(J,V)V=(\nabla_J\nabla_V-\nabla_V\nabla_J-\nabla_{[J,V]})V=\nabla_J\nabla_V V-\nabla_V\nabla_J V$$

$=-\nabla_V\nabla_V J$ (c is a geodesic $\dot{c}=\dfrac{\partial}{\partial t}H$ ，then $\nabla_V V=0$)

$\nabla_V\nabla_V J+R(J,V)V=0$ Jacobi equation

Let $A(t)=\dfrac{D}{dt}\dot{c}$ "acceleration vector field" ，$\dfrac{D}{dt}$ 表示協變微分

$$\frac{D^2}{dt^2}J(t)=\frac{D}{dt}\frac{D}{dt}\frac{\partial}{\partial u}H\Big|_{u=0}=\frac{D}{dt}\frac{D}{du}\frac{\partial}{\partial t}H\Big|_{u=0}$$

$$=\frac{D}{du}\frac{D}{dt}\frac{\partial}{\partial t}H\Big|_{u=0}+R\Big(\frac{\partial H}{\partial t},\frac{\partial H}{\partial u}\Big)\frac{\partial H}{\partial t}\Big|_{u=0}$$

$\nabla_X X=0$ for a geodesic，$R\Big(\dfrac{\partial H}{\partial t},\dfrac{\partial H}{\partial u}\Big)\dfrac{\partial H}{\partial t}\Big|_{u=0}=R(\dot{c}(t),V(t))\dot{c}(t)$

J satisfies the equation $\dfrac{D^2}{dt^2}J(t)=R(\dot{c}(t),V(t))\dot{c}(t)$

J(t) is a Jacobi field along c

定理

設測地線$\gamma(t)$上的一個變分向量場為 Jacobi 場 J，而且 J(0)=J(b)=0，則其相應的二階變分$L''(0)=0$ 。

習作

1. Recall that a vector field J along a geodesic γ is called a Jcobi field if

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0$$

(a) If J is a Jacobi vector field along a geodesic γ ， show that

$g(J(t), \gamma'(t)) = tg(J(0), \gamma'(0))$ for all t

(b) Prove that if $J(t_0) = 0$ for some t_0 ， then $J'(0)$ must be orthogonal to $\gamma'(0)$