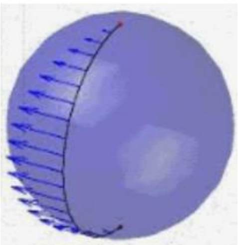
Examples of Jacobi fields

1. Jacobi equation $\frac{D^2 J}{dt} + R(\mathrm{J}, \dot{\gamma})\dot{\gamma} = 0$

Assume $\gamma(t) = (\cos t, \sin t, 0)$ ，a geodesic lied in the equatorial plane。
Consider a variation of $\gamma(t)$ by rotating the sphere slightly around the x-axis。
This gives a family of geodesics： $\gamma_s(t) = (\cos t, \sin t \cos s, \sin t \sin s)$ ，where s is the parameter of the variation。 $\gamma_0(t) = \gamma(t)$
The Jacobi field J(t) is obtained by differentiating $\gamma_s(t)$ with recpect to s at s=0
$J(t) = \frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0}$ ，we have J(t)=(0,-sint,cost)
Interpretation：

- The Jacobi field $J(t) = (0, -\sin t, \cos t)$ describes how nearby geodesics (great circles) deviate from $\gamma(t)$ as s varies.
- At $t = 0$, $J(0) = (0, 0, 1)$, which is perpendicular to $\dot{\gamma}(0) = (0, 1, 0)$. This reflects the fact that the variation is orthogonal to the geodesic at $t = 0$.
- As t increases, $J(t)$ rotates in the plane orthogonal to $\dot{\gamma}(t)$, reflecting the curvature of S^2.

2. $\omega(t)$ 是 γ 上切於緯圓 C_t 的平行向量場，$|\omega| = 1$ ，且 $<\omega(t), \gamma'(t)> = 0$

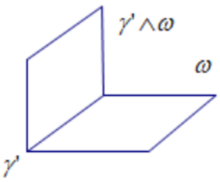


則 $J(t) = (\sin t)\omega(t)$ 是 γ 上的一個 Jacobi 場。

$\frac{dJ}{dt} = (\cos t)\omega(t) + (\sin t)\omega'(t)$ ，$|\omega(t)| = 1, \omega \cdot \omega' = 0$ ，

取切部，則 $\frac{DJ}{dt} = (\cos t)\omega(t)$ ，同理

$\frac{D}{dt}\frac{DJ}{dt} = (-\sin t)\omega(t)$



因為 $\omega(t), \gamma'(t)$ 皆為平行移動，故保持度量，即
$<\omega(t), \gamma'(t)> = 0$
$(\gamma'(t) \wedge \omega(t)) \wedge \gamma'(t) = \omega(t)$
因此 $(\gamma'(t) \wedge J(t)) \wedge \gamma'(t) = J(t)$
R=K=1

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = (-\sin t)\omega(t) + (\sin t)\omega(t) = 0$$

所以 $J(t) = (\sin t)\omega(t)$ 是 γ 上的一個 Jacobi 場。
注意到 $J(\pi) = 0$

3. A simple and explicit example of a Jacobi field on S^3

Let $\gamma(t)=(\cos t,\sin t,0,0)$ is a great circle in the (x_1,x_2)-plane of R^4

Define the variation $\gamma_s(t)=(\cos t,\sin t,s\cos t,s\sin t)$ ，then J(t)=(0,0,cost,sint) is the Jacobi field。

在 S^3 上曲率恆為+1，Jacobi 場的通解在測地線上有著標準形式。由於 S^3 的截面曲率為 1，Jacobi 場的方程可簡化為： $J''(t)+J(t)=0$

其一般解為 J(t)=Acost+Bsin(t)，其中 A,B 是測地線上與 $\dot{\gamma}(t)$ 正交的恆向量。

幾何解釋：

這個 Jacobi 場可以理解為將測地線 $\gamma(t)$ 在不同的「大圓」之間平移的無窮小變化。由於 S^3 是均勻且各向同性的空間，Jacobi 場可以看作是產生等距同構（isometry）族的一部分，這些等距同構將一條大圓測地線變換為另一條。

Properties：

1. **Orthogonality**: The Jacobi field $J(t)=(0,0,\cos t,\sin t)$ is orthogonal to the tangent vector $\dot{\gamma}(t)=(-\sin t,\cos t,0,0)$ of the geodesic $\gamma(t)$. This is a general property of Jacobi fields arising from variations through geodesics.

- The Jacobi field $J(t)=(0,0,\cos t,\sin t)$ describes how nearby geodesics deviate from $\gamma(t)$ as s varies.
- At $t=0$, $J(0)=(0,0,1,0)$, which is perpendicular to $\dot{\gamma}(0)=(0,1,0,0)$. This reflects the fact that the variation is orthogonal to the geodesic at $t=0$.
- As t increases, $J(t)$ rotates in the (x_3,x_4)-plane, reflecting the curvature of S^3.

3. **Behavior of $J(t)$**: The Jacobi field $J(t)=(0,0,\cos t,\sin t)$ oscillates sinusoidally as t increases. This reflects the positive curvature of S^3, which causes nearby geodesics to converge and diverge periodically.