A**Jacobi field** is a concept in Riemannian geometry that arises in the study of geodesics and their variations。It provides important information about the behavior of nearby geodesics and plays a key role in understanding the curvature and topology of Riemannian manifolds。

Definition

Let $\gamma(t)$ be a geodesic on a Riemannian manifold M，and let $\gamma_s(t)$ be a smooth one-parameter family of geodesics such that $\gamma_0(t) = \gamma(t)$ 。

A Jacobi field J(t) along $\gamma(t)$ is the variation vector field of this family，defined as

$$J(t) = \frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0}$$

Jacobi equation $\frac{D^2 J(t)}{dt^2} + R(J, \dot{\gamma})\dot{\gamma} = 0$

Applications：

1. **Conjugate Points**：If a Jacobi field vanishes at two distinct points along a geodesic，these points are called **conjugate points**。The existence of conjugate points is related to the curvature of the manifold。
2. **Comparison Theorems**：Jacobi fields are used in comparison theorems (e.g., Rauch comparison theorem) to compare the geometry of a given manifold with that of a model space (e.g., a sphere or hyperbolic space)。
3. **Index Theory**：Jacobi fields are used in Morse theory to study the topology of the space of geodesics on a manifold。