摘要：

1. Jacobi field equation 的定義 實例
2. Jacobi field 與變分法的關係
3. 如何用 exponential map 建構 Jacobi field
4. Jacobi field 的共軛點有何重要性質

   The most important fact about conjugate points is that they are precisely the images of singularities of the exponential map。

   singularities of the exponential map。(i.e. points where it fails to be a local diffeomorphism。)
5. 了解 Jacobi 場，便可掌握測地線的大域行為，進一步探討 Riemann 流形的整體樣貌。
6. CMC 上的 Jacobi fields
7. Jacobi fields and tidal effects in Kerr Spacetime

§ 測地線 geodesic

$\gamma : I \to M, \gamma(t) = (x_1(t), x_2(t), ..., x_n(t))$ is a geodesic $\Leftrightarrow \frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0$

TM$=\left\{(p,v)\middle| p \in M, v \in T_p M\right\}$ is a tangent bundle ，

{U，x}是 M 的坐標系，在 TU 的座標得到 TM 的 differentiable structure

$p(x_1,...,x_n), v = \sum_i y_i \frac{\partial}{\partial x^i}$ ，則 $(p,v) = (x_1,...,x_n,y_1,...,y_n)$

A geodesic $t \to (x_1(t),...,x_n(t),\frac{dx_1}{dt},...,\frac{dx_n(t)}{dt})$ 滿足 $\begin{cases} \frac{dx_k}{dt} = y_k \\ \frac{dy_k}{dt} = -\Gamma_{ij}^k y_i y_j \end{cases}$

§ Jacobi 場

γ:[0,l] $\to$ M is a geodesic。$T = \frac{d\gamma}{dt}$ ，

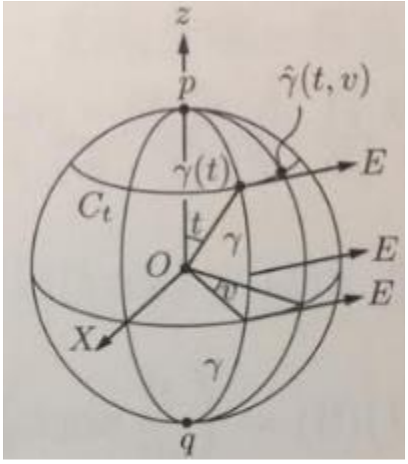
J(t) is a vector field J(t) along γ ，satisfies

$\frac{D^2 J}{dt^2} + R(J,\mathrm{T})\mathrm{T} = 0$ for all $t \in [0,l]$

Then J(t) is called a Jacobi field。

其中 $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ 為曲率張量。

§ 例 [大域微分幾何 p.232]

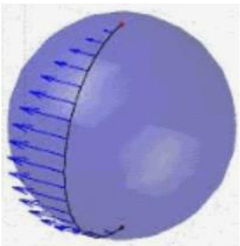


$\gamma(t)$是一經線(geodesic)，$t \in [0,\pi]$

將$\gamma(t)$做變分 $\gamma_v(t) = \hat{\gamma}(t,v)$

使每一條γ_v都是測地線

則$J = \dfrac{\partial \hat{\gamma}}{\partial v}$即$\gamma$上的一個 Jacobi 場。



$\omega(t)$是γ上切於緯圓C_t的平行向量場，$|\omega| = 1$，且

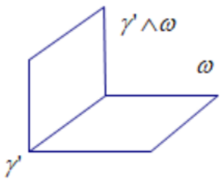
$< \omega(t), \gamma'(t) >= 0$

則$J(t) = (\sin t)\omega(t)$是γ上的一個 Jacobi 場。

$\dfrac{dJ}{dt} = (\cos t)\omega(t) + (\sin t)\omega'(t)$，$|\omega(t)| = 1, \omega \cdot \omega' = 0$，

取切部，則$\dfrac{DJ}{dt} = (\cos t)\omega(t)$，同理

$\dfrac{D}{dt}\dfrac{DJ}{dt} = (-\sin t)\omega(t)$



因為$\omega(t), \gamma'(t)$皆為平行移動，故保持度量，即

$< \omega(t), \gamma'(t) >= 0$

$(\gamma'(t) \wedge \omega(t)) \wedge \gamma'(t) = \omega(t)$

因此 $(\gamma'(t) \wedge J(t)) \wedge \gamma'(t) = J(t)$

R=K=1

$\dfrac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = (-\sin t)\omega(t) + (\sin t)\omega(t) = 0$

所以$J(t) = (\sin t)\omega(t)$是γ上的一個 Jacobi 場。

注意到$J(\pi) = 0$

§ Pro[osition

1. $J_1(t), J_2(t)$是沿著$\gamma:[0,l] \to M$的兩 Jacobi 場 則

$$< \frac{DJ_1}{dt}, J_2(t) > - < J_1(t), \frac{DJ_2(t)}{dt} >= const$$

2. J(t)是沿著$\gamma:[0,l] \to M$的 Jacobi 場 $< J(t), \gamma'(t_1) >=< J(t), \gamma'(t_2). = 0$ for

$t_1 \neq t_2$ 則 $< J(t), \gamma'(t) >= 0, \forall t \in [0,l]$

3. γ 是測地線，給定 J(0)、 $J'(0)$ ，A,B $\in T_P(M), \gamma(0) = P$

則存在唯一一個 γ 上的 Jacobi 場 滿足 J(0)=A， $J'(0) = B$ ，其中 J' 表示 $\nabla_T J$

§ Jacobi 場與潮汐力(Tidal forces)

在廣義相對論中，兩物體開始時沿兩平行軌道(trajectory)運動，潮汐力(tidal force)的存在(時空曲率的存在)會造成軌道接近或遠離，在此兩物體間產生相對加速度。

因此，所謂的測地線偏離效應，它是引力相互作用的一種體現。

數學上，廣義相對論中的潮汐力用黎曼曲率來描述，而一物體在重力影響下的軌道稱為測地線。

測地線偏離方程式描述兩鄰近測地線之間黎曼曲率張量與相對加速度的關係。

或者說，沿測地線 C 的 Jacobi 場的作用在於度量鄰近 C 的測地線偏離 C 的速度。

在微分幾何中，測地線偏離方程式通常稱為 Jacobi 場方程式。

辛格(John Lighton Synge 1897~1995)是最早研究黑洞內部的物理學家之一，
稱 Jacobi fields 方程是測地線偏離(geodesic deviation)方程，在引力波的研究中有應用。

https://kknews.cc/zh-tw/science/v94o9nl.html(盧昌海)
[引力波百年漫談]

§ Relationship between the spreading of geodesics and curvature

Let $p \in M$ and $\gamma : [0,a] \to M$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v$ 。

Let $w \in T_v(T_p M)$ with $|w| = 1$ and J be a Jacobi field along γ given by

$J(t) = (d\exp_p)_{tv}(tw)$ 。Then the Taylor expansion of $|J(t)|^2$ about t=0 is give by

$|J(t)|^2 = t^2 - \frac{1}{3} < R(v,w)v, w > t^4 + R(t)$ ，where $\lim_{t\to 0} \frac{R(t)}{t^4} = 0$

The above equation tells that the geodesic $t \to \exp_p(tv(s))$ deviate from the geodesic

$\gamma(t) = \exp_p tv(0)$ with a velocity that differs from t by a term of third order of t given by

$-\frac{1}{6}K(p,\sigma)t^3$

§ Proposition

If $\gamma:[0,l] \to M$ is parametrized by arc length，and <w ,v>=0 the expression <R(v ,w)v ,w> is the sectional curvature at p with respect to the plane σ generated by v and w。Therefore in this situation $|J(t)|^2 = t^2 - \frac{1}{3}K(p,\sigma)t^4 + R(t)$ ，and

$|J(t)| = t - \frac{1}{6}K(p,\sigma)t^3 + \tilde{R}(t)$ with $\lim_{t\to 0}\frac{\tilde{R}}{t^3} = 0$

§ 如何用 exponential map 建構一個 Jacobi 場

Construct a Jacobi field along a geodesic using the exponential mapping：

$p \in M, v \in T_pM$ such that $\exp_p v$ is defined，

Let $f:[0,1]\times(-\varepsilon,\varepsilon) \to M$ be the parametrized surface given by

$f(t,s) = \exp_p tv(s)$ with v(s) being a curve in T_pM satisfying v(0)=v

$J(t) = \frac{\partial f}{\partial s}(t,0)$ is a vector field along the geodesic $\gamma(t) = \exp_p(tv), 0 \le t \le 1$ ，then J(t) is a Jacobi field。

Q：When is the exponential map a local diffeomorphism？

If (M,g) is complete，we know that $\exp_p$ is defined on all of T_pM ，and is a local diffeomeorphism near 0。

However，it may well happen that it ceases to be even a local diffeomorphism at points far away。

An enlightening example is provided by the sphere S_R^n 。

All geodesics starting at a given point p meet at the antipodal point，which is at a distance of πR along each geodesic。

The exponential map is a diffeomorphism on the ball $B_{\pi R}(0)$ ，but it fails to be a local diffeomorphism at all points on the sphere of radius πR in $T_p S_R^n$ 。

Moreover，each Jacobi field on S_R^n vanishes at p has its first zero precisely at distance πR 。

If U is a normal nbhd of p (the image of a set on which $\exp_p$ is a diffeomorphism)，no Jacobi field that vanishes at p can vanish at any other point。
We might thus to espect a relationship between zeros of Jacobi fields and singularities of the exponential map。(i.e. points where it fails to be a local diffeomorphism。)

§ Conjugate points and the singularities of the exponential map

$\gamma:[0,l]\to M$ 是一測地線，$\gamma(0)=P$ ，$Q=\gamma(t_0)$ $t_0\in[0,l]$

若存在一沿著γ 不全為 0 的 Jacobi 場 J(t)，使得$J(0)=J(t_0)$ ，則稱 Q 是 P 的共軛點(關於測地線γ)。因此，上例中 球面上 Q 是 P 的共軛點。

Let $\gamma:[0,l]\to M$ be a geodesic and $\gamma(0)=p$ 。

The point $q=\gamma(t_0)$ is conjugate to p along γ if and only if $v_0=t_0\gamma'(0)$ s a critical point of $\exp_p$ 。

In addition，the multiplicity of q as a conjugate point of p is equal to the dimension of the kernel of the linear map $(d\exp_p)_{v_0}$ 。

§ And the maximum number of such linearly independent fields is called the multiplicity of the conjugate point to $\gamma(t_0)$

The order of multiplicity of conjugacy is the dimension of the space of Jacobi fields vanishing at p and q。

§ Theorem

Let γ be a smooth timelike curve connecting two points $p,q\in M$ 。
Then the necessary and sufficent condition that γ locally maximizes proper between p and q over smooth one parameter variations is that γ be a geodesic with no point

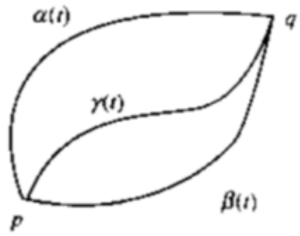
conjugate to p between p and q。
The theorem is the basis for the singularity theorem of GR。

§ Jacobi 場與變分法的關係

Let M is a Riemann manifold with symmetric connection ∇

$c:[a,b]\to M$ a geodesic

$H(t,u):[a,b]\times(-\varepsilon,\varepsilon)\to M$ a variation of c with



H(t,0)=c(t)，H(a,u)=c(a)，H(b,u)=c(b)，and is a geodesic as well

Let $J(t)=\dfrac{\partial H}{\partial u}$ variation vector field (類似切向量)

$\dot{c}(t)=\dfrac{\partial H}{\partial t}$ velocity vector field along c

($J=\dfrac{\partial}{\partial u}, V=\dfrac{\partial}{\partial t}$ ，then $L_J V=[J,V]=0$

Torsion free ，$T(J,V)=\nabla_J V-\nabla_V J-[J,V]=0$ then $\nabla_J V=\nabla_V J$

$$R(J,V)V=(\nabla_J\nabla_V-\nabla_V\nabla_J-\nabla_{[J,V]})V=\nabla_J\nabla_V V-\nabla_V\nabla_J V$$

$$=-\nabla_V\nabla_V J \quad \text{(c is a geodesic } \dot{c}=\frac{\partial}{\partial t}H \text{ ，then } \nabla_V V=0\text{)}$$

$\nabla_V\nabla_V J+R(J,V)V=0$ Jacobi equation

Let $A(t)=\dfrac{D}{dt}\dot{c}$ "acceleration vector field" ，$\dfrac{D}{dt}$ 表示協變微分

$$\frac{D^2}{dt^2}J(t)=\frac{D}{dt}\frac{D}{dt}\frac{\partial}{\partial u}H\big|_{u=0}=\frac{D}{dt}\frac{D}{du}\frac{\partial}{\partial t}H\big|_{u=0}$$

$$=\frac{D}{du}\frac{D}{dt}\frac{\partial}{\partial t}H\big|_{u=0}+R\left(\frac{\partial H}{\partial t},\frac{\partial H}{\partial u}\right)\frac{\partial H}{\partial t}\big|_{u=0}$$

$\nabla_X X=0$ for a geodesic，$R\left(\dfrac{\partial H}{\partial t},\dfrac{\partial H}{\partial u}\right)\dfrac{\partial H}{\partial t}\big|_{u=0}=R(\dot{c}(t),V(t))\dot{c}(t)$

J satisfies the equation $\dfrac{D^2}{dt^2}J(t)=R(\dot{c}(t),V(t))\dot{c}(t)$

J(t) is a Jacobi field along c

定理

設測地線$\gamma(t)$上的一個變分向量場為 Jacobi 場 J，而且 J(0)=J(b)=0，則其相應的二階變分$L''(0)=0$ 。

習作

1. Recall that a vector field J along a geodesic γ is called a Jcobi field if

$$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0$$

(a) If J is a Jacobi vector field along a geodesic γ，show that

$g(J(t),\gamma'(t)) = tg(J(0),\gamma'(0))$ for all t

(b) Prove that if $J(t_0) = 0$ for some t_0，then $J'(0)$ must be orthogonal to $\gamma'(0)$

[DG01 p.116 EX4.8]

§ Jacobi fields and tidal effects in Kerr Spacetime

§ Applications of Jacobi fields to Normal Contact Lorentzian Manifolds

1. General Relativity by Wald 1984
2. The Large Scale Structure of Spacetime by Hawking & Ellis 1973

[大域微分幾何 Ch30]CMC 上的 Jacobi 場與 Morse Index 定理

1. CMC(常均曲率)曲面上 Jacobi 場的分佈
2. Morse Index 定理
3. Geodesics 的穩定
4. Jacobi 場的 multiplicity 分佈 共軛邊界
5. Sobolev Theory
6. 值譜分解